

Baxter operators for arbitrary spin II

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Abstract

This paper presents the second part of our study devoted to the construction of Baxter operators for the homogeneous closed XXX spin chain with the quantum space carrying infinite or finite-dimensional sl_2 representations. We consider the Baxter operators used in [8, 9], formulate their construction uniformly with the construction of our previous paper. The building blocks of all global chain operators are derived from the general Yang-Baxter operators and all operator relations are derived from general Yang-Baxter relations. This leads naturally to the comparison of both constructions and allows to connect closely the treatment of the cases of infinite-dimensional representation of generic spin and finite-dimensional representations of integer or half-integer spin. We prove not only the relations between the operators but present also their explicit forms and expressions for their action on polynomials representing the quantum states.

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1 Introduction

In our previous paper [7]¹ we have described in detail an approach to the construction of Q-operators for the spin chain with the symmetry algebra sl_2 . There exists another approach to the same problem [8, 9]. The aim of the present paper is to unify both approaches in the framework of the Quantum Inverse Scattering Method (QISM) [1–4] and to establish explicit interrelations. We choose the homogeneous closed XXX spin chain as the basic example in particular because the spin $\frac{1}{2}$ case of such a chain has been considered in [9].

The general strategy is the same in both approaches to Baxter Q-operators and it appears universal in the framework of QISM. Formulated for the finite-dimensional sl_2 representation case, the two main steps of the construction are:

- Construction of the general transfer matrix, here for spin of quantum spaces $\ell = \frac{n}{2}, n \in \mathbb{N}$, spin of auxiliary space $s \in \mathbb{C}$ and regularization parameter $q, |q| < 1$ as

$$\mathbf{T}_s(u) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}(u|\frac{n}{2}, s) \mathbf{R}_{20}(u|\frac{n}{2}, s) \cdots \mathbf{R}_{N0}(u|\frac{n}{2}, s) . \quad (1.1)$$

where $\mathbf{R}_{k0}(u|\frac{n}{2}, s)$ is the general solution of the Yang-Baxter equation acting in the tensor product $\mathbb{V}_n \otimes \mathbb{U}_{-s}$, where \mathbb{V}_n is $(n+1)$ -dimensional and \mathbb{U}_{-s} is infinite-dimensional irreducible sl_2 -modules. In the case of half-integer spin $\ell = \frac{n}{2}$ the trace over infinite-dimensional space diverges so that we need the regularization depending on q .

- Proof of the following factorization

$$\mathbf{S} \mathbf{T}_s(u) = \mathbf{Q}_B(u-s) \mathbf{Q}_A(u+s+1) = \mathbf{Q}_A(u+s+1) \mathbf{Q}_B(u-s) , \quad (1.2)$$

where commuting operators \mathbf{Q}_A and \mathbf{Q}_B are some transfer matrices

$$\mathbf{Q}_A(u) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^A(u) \cdots \mathbf{R}_{N0}^A(u) \quad ; \quad \mathbf{Q}_B(u) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^B(u) \cdots \mathbf{R}_{N0}^B(u) ,$$

constructed from the local operators $\mathbf{R}_{k0}^A(u)$ and $\mathbf{R}_{k0}^B(u)$. These local operators are closely related to the general Yang-Baxter operator $\mathbf{R}_{k0}(u|\frac{n}{2}, s)$ and can be seen as building blocks of the latter because of factorization. The operator \mathbf{S} does not depend on spectral parameter u and commutes with $\mathbf{T}_s(u)$ and $\mathbf{Q}_A, \mathbf{Q}_B$.

The factorization (1.2) immediately leads to the determinant representation for the transfer matrix $t_m(u)$ constructed with $(m+1)$ -dimensional auxiliary space. The determinant representation is the origin of the Baxter equation for operators \mathbf{Q}_A and \mathbf{Q}_B and of various fusion relations. We use the general labels A and B , $A, B = +, -$ or $A, B = 1, 2$ because there are two different pairs of Q-operators, \mathbf{Q}_1 and \mathbf{Q}_2 studied in part I and \mathbf{Q}_+ and \mathbf{Q}_- studied in [9].

We have formulated the two basic steps for the case of integer or half-integer spin at the chain sites, $\ell = \frac{n}{2}$, because this is the case on which the studies in [8, 9] are focussed. The steps are analogous in the case of generic spin $\ell \in \mathbb{C}$.

In part I we have treated first the generic spin case and after this we have considered the case of integer values of 2ℓ in analogy. We have been able to give the relations between the global chain operators, the general transfer matrix, the Baxter Q-operators in particular, for the generic and the half-integer spin cases by explicit expressions.

In the present paper we shall formulate the operators appearing in [8, 9] in such a way that their treatment becomes unified with the one of part I. The main goal of this paper is the explicit description of the relations between the two pairs of Q-operators.

¹In the following [7] will be referred to as part I.

Unlike [9] we prefer to consider the generic spin case first and to study from this viewpoint the integer case $2\ell = n$ as a specific case. Recall that we denote by L-matrix an operator acting the tensor product of sl_2 representation spaces where one factor of them is just the fundamental (spin $\frac{1}{2}$) representation, whereas the general Yang-Baxter R operator acts on the tensor product with both factors carrying generic representations. We use the latter as the most general local chain operator, whereas the L-matrix appears as the main building block in [9]. Although the general R operator is derived from the L-matrix (by solution of the *RLL* relation) we have demonstrated in part I the advantages of the construction relying on the former.

The presentation is organized as follows. We start from consideration of the simplest example of the spin one half, treated in [9]. Therefore in section 2 we specify our general formulae for arbitrary ℓ from part I to the case $\ell = \frac{1}{2}$ and give some sketch of the construction of \mathbf{Q}_{\pm} -operators of [9]. By expressing relations in maximally explicit form we illustrate the relations between both constructions before addressing the systematic treatment of the general case.

Section 3 is devoted to local objects and their relations. It is well known [10–12] that matrices L^{\pm} serving as local building blocks for the construction of \mathbf{Q}_{\pm} -operators can be obtained as special limits of the standard L-operator. Using the same limits in the case of general R-matrix one obtains operators R^{\pm} , which are the natural building blocks for construction of \mathbf{Q}_{\pm} -operators if one intends to lift the whole construction to the case of generic spin ℓ . We establish the explicit connection between the two sets of building blocks, the operators R^1 and R^2 from part I and the operators R^{\pm} appearing by lifting L^{\pm} .

Next we follow step by step the same way as in part I. The key local relations needed for the factorization of transfer matrix and commutativity are obtained by applying the mentioned limiting procedure to appropriate relations of part I. We consider the generic spin case in section 4 and then turn to the case of integer and half-integer spin in section 5. In particular the action of the operators on polynomials is written explicitly.

Having derived all necessary formulae for both pairs of Q-operators it is easy to compare them and to discuss their advantages and drawbacks.

2 Spin $\frac{1}{2}$ chain

In this section we specify to the simplest example of two-dimensional representations at each site of the spin chain. In this case the general operator $\mathbf{R}(u|\frac{1}{2}, s)$ acting in the tensor product $\mathbb{C}^2 \otimes \mathbb{U}_{-s}$ is simple and coincides up to normalization and a shift of the spectral parameter with the L-operator

$$\mathbf{R}(u|\frac{1}{2}, s) = -\frac{\Gamma(-s - \frac{1}{2} + u)}{\Gamma(-s + \frac{1}{2} - u)} \cdot L(u + \frac{1}{2}) \quad ; \quad L(u) = u + \vec{\sigma} \otimes \vec{S} = \begin{pmatrix} u + S & S_- \\ S_+ & u - S \end{pmatrix}, \quad (2.1)$$

\vec{S} are generators of sl_2 , $S_{\pm} = S_1 \pm iS_2$ and $S = S_3$, in the infinite-dimensional representation \mathbb{U}_{-s} labeled by spin $s \in \mathbb{C}$. The usual model for this representation is the space of polynomials $\mathbb{C}[z]$ where the generators are realized as first order differential operators

$$S = z\partial - s, \quad S_- = -\partial, \quad S_+ = z^2\partial - 2sz. \quad (2.2)$$

Allowing for a change of normalization and a shift of the spectral parameter, we take $L(u)$ as the building block for the construction of the transfer matrix

$$\mathbf{T}_s(u|q) = \text{tr}_z q^{z\partial} \left(u + \vec{\sigma}_1 \otimes \vec{S} \right) \left(u + \vec{\sigma}_2 \otimes \vec{S} \right) \cdots \left(u + \vec{\sigma}_N \otimes \vec{S} \right). \quad (2.3)$$

Here we use explicit notations for visualization: the quantum space of the chain is the tensor product of two-dimensional spaces $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ and the σ -matrices $\vec{\sigma}_k$ act in \mathbb{C}^2 at site k . The operators \vec{S} act in the auxiliary space $\mathbb{C}[z]$ where also the trace tr_z is calculated.

In this section we outline the construction and basic properties of \mathbf{Q} operators for the example of the spin $\frac{1}{2}$ chain. We start with a sketch of the construction of the paper [9]. Further we present shortly the specification to this example of our construction of part I and give a first discussion of the connection between both constructions.

There exists the useful factorized representation of the operator $L(u)$

$$L(u) = L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \quad u_1 \equiv u - s - 1, u_2 \equiv u + s. \quad (2.4)$$

We have introduced the parameters u_1 and u_2 along with u and s because they are very convenient for our purposes. L -operator respects the Yang-Baxter equation in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{U}_{-s}$ with Yang's R -matrix,

$$R_{ij,nm}(u-v) \cdot L_{nr}(u) \cdot L_{mp}(v) = L_{ir}(v) \cdot L_{jp}(u) \cdot R_{rp,nm}(u-v), \quad (2.5)$$

where $i, j, \dots = 1, 2$ and $R_{ij,nm}(u) = u \cdot \delta_{in} \delta_{jm} + \delta_{im} \delta_{jn}$. Besides of the operator (2.4) it is also useful to consider simpler operators [9, 11, 12]

$$L^+(u) = u e^+ + \mathbf{e} \otimes \mathbf{A}^+ = \begin{pmatrix} u + \partial z & -\partial \\ -z & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u & 0 \\ -z & 1 \end{pmatrix}, \quad (2.6)$$

$$L^-(u) = u e^- + \mathbf{e} \otimes \mathbf{A}^- = \begin{pmatrix} 1 & -\partial \\ z & u - z\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & u \end{pmatrix} \cdot \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}. \quad (2.7)$$

We use explicit notations $e^+ = e_{11}$, $e^- = e_{22}$ where e_{ik} is the standard basis in the space of two-dimensional matrices and $\mathbf{e} \otimes \mathbf{A} = e_{11} A_{11} + e_{12} A_{12} + e_{21} A_{21} + e_{22} A_{22} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. We did this to exhibit clearly the separation of the action in quantum space by matrices from the action in auxiliary space by operators acting on polynomials in z . The matrix elements A_{ik}^\pm are operators on \mathbb{U}_{-s} like S_\pm and S . In analogy with (2.3) we consider the transfer matrices

$$\mathbf{Q}_\pm(u) = \text{tr}_z q^{z\partial} (u e_1^\pm + \mathbf{e}_1 \otimes \mathbf{A}^\pm) (u e_2^\pm + \mathbf{e}_2 \otimes \mathbf{A}^\pm) \cdots (u e_N^\pm + \mathbf{e}_N \otimes \mathbf{A}^\pm). \quad (2.8)$$

The operators L^\pm also satisfy the Yangian relation (2.5) as well as almost trivial operator $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. The Yangian algebra possesses the important property of co-multiplication. Let $L_1(u)$ and $L_2(u)$ be two solutions of (2.5), acting in the spaces $\mathbb{C}[z_1]$ and $\mathbb{C}[z_2]$ correspondingly. Then the matrix $L_1(u + \delta_1) L_2(u + \delta_2)$ is also a solution of (2.5) acting in the space $\mathbb{C}[z_1, z_2] = \mathbb{C}[z_1] \otimes \mathbb{C}[z_2]$. The possibility to multiply solutions provides an opportunity to construct many solutions starting from simpler ones. In particular it is possible to construct two more complex solutions of (2.5) out of the above simpler ones: $L_1^-(u_2) L_2^+(u_1)$ and $\begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} L_2(u_1, u_2)$. As was pointed out to us by V. Tarasov the general theory developed in the papers [13] implies the existence of an intertwining operator for such two representations of the Yangian algebra. In this particular case the intertwining operator $e^{z_2 \partial_1}$ can be exhibited explicitly

$$e^{z_2 \partial_1} \cdot L_1^-(u_2) L_2^+(u_1) = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} L_2(u_1, u_2) \cdot e^{z_2 \partial_1}. \quad (2.9)$$

This can be checked by

$$e^{z_2 \partial_1} \begin{pmatrix} 1 & 0 \\ z_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 & -\partial_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\partial_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ -z_2 & 1 \end{pmatrix} e^{-z_2 \partial_1} =$$

$$= \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ z_2 & u_2 \end{pmatrix} \begin{pmatrix} 1 & -\partial_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ -z_2 & 1 \end{pmatrix}$$

performing similarity transformation in the left hand side for each matrix factor separately. This local factorization relation (2.9) is a cornerstone of the construction.

Similarly one can consider another pair of solutions: $L_1^+(u_1) L_2^-(u_2)$ and $\begin{pmatrix} 1 & -\partial_1 \\ 0 & 1 \end{pmatrix} L_2(u_1, u_2)$. Again, these representations of Yangian algebra must be equivalent which implies the existence of the intertwining operator r .

$$r \cdot L_1^+(u_1) L_2^-(u_2) = \begin{pmatrix} 1 & -\partial_1 \\ 0 & 1 \end{pmatrix} L_2(u_1, u_2) \cdot r. \quad (2.10)$$

It can be checked by explicit calculation that the expression for intertwining operator is

$$r = \Gamma(z_2 \partial_2 + u_1 - u_2 + 1) e^{z_1 \partial_2}.$$

Following the standard argument of the proof of the commutativity of the transfer matrix (recalled in part I and applied repeatedly there) these intertwining relations lead in the next step to corresponding global factorization relations for chain operators. These relations involve the transfer matrices $\mathbf{T}_s, \mathbf{Q}_\pm$ acting in the whole quantum space of the chain. The first local relation (2.9) leads to

$$\frac{1}{1-q} \mathbf{T}_s(u|q) = \mathbf{Q}_-(u+s) \mathbf{Q}_+(u-s-1) \quad (2.11)$$

while the second local factorization relation (2.10) produces

$$\frac{1}{1-q} \mathbf{T}_s(u|q) = \mathbf{Q}_+(u-s-1) \mathbf{Q}_-(u+s) \quad (2.12)$$

As one can see the q -regularization is indispensable in order to ensure converge of the traces over the infinite dimensional spaces. From (2.11) and (2.12) we deduce commutativity

$$[\mathbf{Q}_-(u), \mathbf{Q}_+(v)] = 0$$

In order to prove the other commutativity relations,

$$[\mathbf{Q}_-(u), \mathbf{Q}_-(v)] = [\mathbf{Q}_+(u), \mathbf{Q}_+(v)] = 0,$$

one has to resort to the local intertwining relations [11,12],

$$P_{12} (1 - z_2 \partial_1)^{u-v} \cdot L_1^-(u) L_2^-(v) = L_2^-(v) L_1^-(u) \cdot P_{12} (1 - z_2 \partial_1)^{u-v},$$

and

$$P_{12} (1 + z_1 \partial_2)^{u-v} \cdot L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) \cdot P_{12} (1 + z_1 \partial_2)^{u-v}.$$

Thus all operators commute

$$[\mathbf{T}_s(u|q), \mathbf{Q}_-(v)] = [\mathbf{T}_s(u|q), \mathbf{Q}_+(v)] = 0.$$

As the consequence of the factorizations (2.11) and (2.12) one obtains the Baxter equations for operators \mathbf{Q}_\pm

$$t(u|q) \mathbf{Q}_-(u) = u^N \mathbf{Q}_-(u+1) + q(u+1)^N \mathbf{Q}_-(u-1), \quad (2.13)$$

$$t(u|q) \mathbf{Q}_+(u) = qu^N \mathbf{Q}_+(u+1) + (u+1)^N \mathbf{Q}_+(u-1), \quad (2.14)$$

where in the transfer matrix $t(u|q)$ the trace is taken over two-dimensional auxiliary space,

$$t(u|q) = \text{tr} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} L_1(u) L_2(u) \cdots L_N(u).$$

Explicit formulae for the Baxter operators \mathbf{Q}_{\pm} are presented in section 5.6.

Now we specify our formulae from part I for the considered case $\ell = \frac{1}{2}$ allowing for some change of normalizations for simplicity. We start again from (2.1) relating the general operator $\mathbf{R}(u|\frac{1}{2}, s)$ acting in the tensor product $\mathbb{C}^2 \otimes \mathbb{U}_{-s}$ with the L-operator. Next we rewrite the expression for $\mathbf{R}(u - v|\frac{1}{2}, s)$ using the parametrization

$$u_1 = u - \frac{3}{2}, \quad u_2 = u + \frac{1}{2}; \quad v_1 = v - s - 1, \quad v_2 = v + s,$$

as

$$\mathbf{R}(u_1, u_2|v_1, v_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_2 - v_2 - 1 & -\partial \\ 0 & u_1 - v_1 + 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix},$$

and consider the limits $v_2 \rightarrow u_2$

$$\mathbf{R}(u_1, u_2|v_1, u_2) = \mathbf{R}^1(u - v_1) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} -1 & -\partial \\ 0 & u_1 - v_1 + 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \quad (2.15)$$

and $v_1 \rightarrow u_1$

$$\mathbf{R}(u_1, u_2|u_1, v_2) = \mathbf{R}^2(u - v_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_2 - v_2 - 1 & -\partial \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}. \quad (2.16)$$

In analogy we consider transfer matrices $\mathbf{Q}_1, \mathbf{Q}_2$ constructed with $\mathbf{R}^1, \mathbf{R}^2$. Finally, as a building block for a simple type of transfer matrix denoted by \mathbf{S} we use the result of the double limit $v_1 \rightarrow u_1$ and $v_2 \rightarrow u_2$

$$\mathbf{R}(u_1, u_2|u_1, u_2) = \mathbf{S}_{12} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} -1 & -\partial \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}.$$

Our counterpart of the factorization (2.11, 2.12) looks as follows (compare part I, section 5)

$$\mathbf{S} \mathbf{T}_s(u) = \mathbf{Q}_2(u - s) \mathbf{Q}_1(u + s + 1) = \mathbf{Q}_1(u + s + 1) \mathbf{Q}_2(u - s), \quad (2.17)$$

where

$$\begin{aligned} \mathbf{Q}_1(u) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^1(u) \cdots \mathbf{R}_{N0}^1(u), \\ \mathbf{Q}_2(u) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^2(u) \cdots \mathbf{R}_{N0}^2(u), \\ \mathbf{S} &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{S}_{10} \mathbf{S}_{20} \cdots \mathbf{S}_{N0}. \end{aligned} \quad (2.18)$$

Of course, the two sets of Q-operators should be connected. To establish this connection we have another look at the definition of transfer matrix and its factorization to \mathbf{Q}_{\pm} -operators,

$$\begin{aligned} \mathbf{T}_s(u - v) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbf{R}_{N0}(u_1, u_2|v_1, v_2), \\ \frac{1}{1 - q} \mathbf{T}_s(u - v) &= \mathbf{Q}_-(u - v + s) \mathbf{Q}_+(u - v - s - 1). \end{aligned}$$

Specifying parameters in the first formula we reduce the transfer matrix $\mathbf{T}_s(u - v)$ to the operator \mathbf{Q}_1 for $v_2 = u_2 \leftrightarrow s = u - v + \frac{1}{2}$ and to the operator \mathbf{Q}_2 for $v_1 = u_1 \leftrightarrow s = v - u + \frac{1}{2}$. On the other hand the substitution of these values of parameter s in the factorization formula results in an expression in terms of \mathbf{Q}_{\pm} -operators. After all one obtains

$$\mathbf{Q}_1(u + 1) = (1 - q) \mathbf{Q}_+(-\frac{3}{2}) \cdot \mathbf{Q}_-(u) \quad ; \quad \mathbf{Q}_2(u + 1) = (1 - q) \mathbf{Q}_-(\frac{1}{2}) \cdot \mathbf{Q}_+(u)$$

The operators $\mathbf{Q}_+(-\frac{3}{2})$ and $\mathbf{Q}_-(\frac{1}{2})$ do not depend on spectral parameter u and commute with the others. There are various forms of such interrelations as we discuss below.

3 Local operators

In this section we start the systematic consideration working out the general construction as outlined in Introduction. We consider first the operators representing the local building units of the chains. Actually the L-operator contains the local information about the system and the \mathbb{R} -operators are derived therefrom. On the other hand the \mathbb{R} -operators are more convenient as building units acting on the tensor product of quantum and auxiliary spaces carrying arbitrary representations. The particular case of spin $\frac{1}{2}$ representation in one of these spaces leads us back to the L-matrix. The \mathbb{R} -operators provide us with the starting point from which the different versions of Baxter operators can be obtained. At first we discuss the operators L^\pm introduced above and then consider their analogues \mathbb{R}^\pm on the level of R-operators.

3.1 L-operators

The sl_2 Lie algebra generators (2.2) appear in the evaluation representations of the Yangian algebra, which is generated by the matrix elements of the L-operator with the fundamental *RLL* relation (2.5) being the algebra relations. Besides of the considered standard ones there are other Yangian representations, described by L-operators obeying (2.5) but different from (2.4). These can be obtained from the ordinary L-operators in the limits $u_2 \rightarrow \infty$ or $u_1 \rightarrow \infty$ and will be denoted by L^+ (2.6) and L^- (2.7) correspondingly. Below we present the necessary calculations for completing the picture. The operators L^\pm appear in the degenerate self-trapping (DST) chain model [11, 12]. On the other hand the ordinary L-operator can be reconstructed from the degenerate ones L^\pm by means of the formula (2.9). For our purposes it is important that both the reduction and the reconstruction relations can be lifted from the L-operators to the \mathbb{R} -operator level.

The dependence of the L-operator on the two parameters u_1, u_2 can be factorized in two ways,

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & u_2 \end{pmatrix} \begin{pmatrix} u_1 + \partial z & -\partial \\ -z & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\partial \\ z & u_2 - z\partial \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ -z & 1 \end{pmatrix}. \quad (3.1)$$

Let us rewrite this taking into account the notations (2.6) , (2.7)

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & u_2 \end{pmatrix} \cdot L^+(u_1) = L^-(u_2) \cdot \begin{pmatrix} u_1 & 0 \\ -z & 1 \end{pmatrix}. \quad (3.2)$$

Using the factorization it is easy to derive what happens with the L-operator in the limit $u_2 \rightarrow \infty$

$$\begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix}^{-1} L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ \frac{z}{u_2} & 1 \end{pmatrix} \cdot L^+(u_1) \longrightarrow L^+(u_1) \quad (3.3)$$

and similarly in the limit $u_1 \rightarrow \infty$

$$L(u_1, u_2) \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = L^-(u_2) \begin{pmatrix} 1 & 0 \\ -\frac{z}{u_1} & 1 \end{pmatrix} \longrightarrow L^-(u_2) \quad (3.4)$$

so that asymptotically we have

$$L(u_1, u_2) \xrightarrow{u_1 \rightarrow \infty} L^-(u_2) \cdot \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad L(u_1, u_2) \xrightarrow{u_2 \rightarrow \infty} \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} \cdot L^+(u_1). \quad (3.5)$$

3.2 General R-operators

Let us recall some notations from part I. The general R-operator acts on the space $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$ and is defined as the sl_2 -invariant solution of the following relation (RLL-relation)

$$R_{12}(u_1, u_2|v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R_{12}(u_1, u_2|v_1, v_2) \quad (3.6)$$

where

$$u_1 = u - \ell_1 - 1, \quad u_2 = u + \ell_1; \quad v_1 = v - \ell_2 - 1, \quad v_2 = v + \ell_2. \quad (3.7)$$

It is also of interest to define the operators R_{12}^1 and R_{12}^2 acting in the space $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$ by the following relations

$$R_{12}^1 L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, u_2) L_2(u_1, v_2) R_{12}^1, \quad (3.8)$$

$$R_{12}^2 L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_2) L_2(v_1, u_2) R_{12}^2. \quad (3.9)$$

Upper indices 1, 2 distinguish our two operators and lower indices as usually shows in what spaces the operators act nontrivially. In generic situation $\ell_1, \ell_2 \in \mathbb{C}$ the solutions of the defining equations (3.8, 3.9) have the form

$$\begin{aligned} R_{12}^1(u_1|v_1, v_2) &= \frac{\Gamma(z_{21}\partial_2 + u_1 - v_2 + 1)}{\Gamma(z_{21}\partial_2 + v_1 - v_2 + 1)}, \\ R_{12}^2(u_1, u_2|v_2) &= \frac{\Gamma(z_{12}\partial_1 + u_1 - v_2 + 1)}{\Gamma(z_{12}\partial_1 + u_1 - u_2 + 1)}. \end{aligned} \quad (3.10)$$

The general R-operator is factorized as follows (see part I)

$$R(u_1, u_2|v_1, v_2) = R^1(u_1|v_1, u_2) R^2(u_1, u_2|v_2) = R^2(v_1, u_2|v_2) R^1(u_1|v_1, v_2). \quad (3.11)$$

Our purpose is to construct transfer matrices for arbitrary spin in quantum space. For this we need appropriate local building blocks. In the previous section we have obtained explicit expressions for L^\pm by means of specific limiting procedure. Now we going to present explicit expressions for operators \mathbb{R}^\pm , the lift of L^\pm to the R-operator level. For this purpose we consider the limiting procedure in RLL-relations and obtain the degenerate R-operators as the leading asymptotics of $R(u_1, u_2|v_1, v_2)$ at large values of one or several of its parameters u_1, u_2, v_1, v_2 like in (3.5). In this section we do not present the corresponding calculations being rather simple but technical ones and collect final results only. All details can be found in Appendix A. Throughout of this section we assume that the spin parameter in quantum space is a generic complex number. The case of (half)-integer quantum spin is considered in Section 5.2.

3.2.1 Reduction $R^1, R^2 \rightarrow r^+, r^-$

By taking the limit $v_1 \rightarrow \infty$ in the defining equation for the operator R^1 (3.8) it is not difficult to deduce

$$r^+(u_1|v_2) \cdot L_1(u_1, u_2) L_2^-(v_2) = L_1^-(u_2) L_2(u_1, v_2) \cdot r^+(u_1|v_2) \quad (3.12)$$

$$v_1^{z_2\partial_2} R^1(u_1|v_1, v_2) \rightarrow r^+(u_1|v_2) = \Gamma(z_2\partial_2 + u_1 - v_2 + 1) e^{z_1\partial_2} \quad (3.13)$$

In analogy with (3.5) we have cancelled out the divergent dilatation operator from the expression of the operator R^1 . Similarly from (3.9) one obtains the reduction $v_2 \rightarrow \infty$

$$r^-(u_1|u_2) \cdot L_1(u_1, u_2) L_2^+(v_1) = L_1^+(u_1) L_2(v_1, u_2) \cdot r^-(u_1|u_2), \quad (3.14)$$

$$R^2(u_1, u_2|v_2) v_2^{-z_1\partial_1} \rightarrow r^-(u_1|u_2) = e^{-z_2\partial_1} \frac{(-)^{z_1\partial_1}}{\Gamma(z_1\partial_1 + u_1 - u_2 + 1)} \quad (3.15)$$

3.2.2 Reduction $R \rightarrow R^+, R^-$

The reduction in the defining equation of the R -operator (3.6) at $v_1 \rightarrow \infty$ leads to

$$R^+(u_1, u_2|v_2) \cdot L_1(u_1, u_2) L_2^-(v_2) = L_1^-(v_2) L_2(u_1, u_2) \cdot R^+(u_1, u_2|v_2) \quad (3.16)$$

$$v_1^{z_2 \partial_2} R(u_1, u_2|v_1, v_2) \rightarrow R^+(u_1, u_2|v_2) = r^+(u_1|u_2) \cdot R^2(u_1, u_2|v_2) \quad (3.17)$$

and at $v_2 \rightarrow \infty$

$$R^-(u_1, u_2|v_1) \cdot L_1(u_1, u_2) L_2^+(v_1) = L_1^+(v_1) L_2(u_1, u_2) \cdot R^-(u_1, u_2|v_1) \quad (3.18)$$

$$R(u_1, u_2|v_1, v_2) v_2^{-z_1 \partial_1} \rightarrow R^-(u_1, u_2|v_1) = R^1(u_1|v_1, u_2) \cdot r^-(u_1|u_2) \quad (3.19)$$

Consequently we have to our disposal the following factorization formulae: (3.11) for the general R -operator and (3.17), (3.19) for its two reductions R^+, R^- . In fact factorization relations for the reduced operators are direct consequence of the factorization relations for the general R -operator. These formulae have analogues on the level of transfer matrices as we will see soon. Now we have the appropriate local building blocks to be implemented in the construction of different transfer matrices.

4 Global objects: commuting transfer matrices

In the previous section we have introduced the local operators which concern only one site of the spin chain. Now we turn to the description of the whole system. We are going to construct various families of commuting operators – transfer matrices and Q -operators.

Let us recall some notations from part I. We distinguish two versions of the general Yang-Baxter operators acting in the space $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$ by the notations R_{12} and \mathbb{R}_{12} . The former does not contain the permutation operator P_{12} whereas the latter does, so they are related by

$$\mathbb{R}_{12} = P_{12} R_{12}.$$

We will also use the notions $r^\pm, \mathbb{R}^\pm, R^1$ and \mathbb{R}^2 which are related to r^\pm, R^\pm, R^1 and R^2 in a similar way. The general transfer matrix $T(u)$ for the homogeneous XXX-spin chain is constructed from the local operators $\mathbb{R}_{k0}(u|\ell, s)$ acting in the tensor product of quantum space $\mathbb{U}_{-\ell}$ and auxiliary space \mathbb{U}_{-s} . The trace is taken over the generic infinite-dimensional auxiliary space

$$T_s(u|q) = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbb{R}_{10}(u|\ell, s) \mathbb{R}_{20}(u|\ell, s) \cdots \mathbb{R}_{N0}(u|\ell, s) \right]. \quad (4.1)$$

At fixed spin ℓ the free parameters in the general transfer-matrix $T_s(u|q)$ are the spectral parameter u and the spin parameter s in the auxiliary space. We recall the relation to our four-parameter notation (3.7)

$$\begin{aligned} \mathbb{R}_{k0}(u - v|\ell, s) &= \mathbb{R}_{k0}(u_1, u_2|v_1, v_2), \\ u_1 &= u - \ell - 1, u_2 = u + \ell, v_1 = v - s - 1, v_2 = v + s. \end{aligned} \quad (4.2)$$

In this notation the above definition can be rewritten as

$$T_s(u - v|q) = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbb{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2) \right]. \quad (4.3)$$

We assume that the spin in the quantum space ℓ is generic. As we have explained in part I this definition is suited for both finite-dimensional and infinite-dimensional representations in the quantum space. If the quantum spin parameter ℓ is generic, $\ell \in \mathbb{C}, 2\ell + 1 \notin \mathbb{N}$, the trace in (4.1) is

convergent even without q -regularization. However we shall see shortly that such regularization is unavoidable in the construction of the operators Q_{\pm} for all values of ℓ .

Our next goal is to obtain the factorization of the general transfer matrix (4.1) into the product of two other transfer matrices built from simpler local blocks and to prove commutativity properties for the general transfer matrix and its two factors. We have carried out this program in part I using local the three-term relations. They serve for our current purposes as well. Let us quote them here again. At first we have the general Yang-Baxter equation involving three general \mathbb{R} -operators

$$\begin{aligned} \mathbb{R}_{23}(v_1, v_2|w_1, w_2)\mathbb{R}_{13}(u_1, u_2|w_1, w_2)\mathbb{R}_{12}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{12}(u_1, u_2|v_1, v_2)\mathbb{R}_{13}(u_1, u_2|w_1, w_2)\mathbb{R}_{23}(v_1, v_2|w_1, w_2) \end{aligned} \quad (4.4)$$

Then we have the relation involving one operator \mathbb{R}^1 and two \mathbb{R} -operators

$$\begin{aligned} \mathbb{R}_{23}^1(v_1|w_1, w_2)\mathbb{R}_{13}(u_1, u_2|w_1, w_2)\mathbb{R}_{12}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{12}(u_1, u_2|v_1, v_2)\mathbb{R}_{13}(u_1, u_2|w_1, w_2)\mathbb{R}_{23}^1(v_1|w_1, w_2) \end{aligned} \quad (4.5)$$

and finally the relation with one operator \mathbb{R}^2 and two \mathbb{R} -operators

$$\begin{aligned} \mathbb{R}_{23}^2(v_1, v_2|w_2)\mathbb{R}_{13}(u_1, u_2|w_1, w_2)\mathbb{R}_{12}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{12}(u_1, u_2|w_1, v_2)\mathbb{R}_{13}(u_1, u_2|v_1, w_2)\mathbb{R}_{23}^2(v_1, v_2|w_2). \end{aligned} \quad (4.6)$$

Starting from these three relations (4),(4.5),(4.6) it is possible to derive the factorization of the general transfer matrix $T(u)$ into the product of Q -operators, commutativity of all these operators and also to obtain the Baxter equation.

4.1 Factorization and commutativity of the general transfer matrix

The general transfer matrix has the remarkable factorization properties

$$\frac{1}{1-q} \cdot T_s(u-v|q) = Q_+(u-v_2) Q_-(u-v_1) = Q_-(u-v_1) Q_+(u-v_2). \quad (4.7)$$

where the operators Q_+ and Q_- are transfer matrices constructed from \mathbb{R}^+ and \mathbb{R}^-

$$Q_-(u-v_1) = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbb{R}_{10}^-(u_1, u_2|v_1) \cdots \mathbb{R}_{N0}^-(u_1, u_2|v_1) \right], \quad (4.8)$$

$$Q_+(u-v_2) = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbb{R}_{10}^+(u_1, u_2|v_2) \cdots \mathbb{R}_{N0}^+(u_1, u_2|v_2) \right]. \quad (4.9)$$

Notice that the dependence on the parameters v_1 and v_2 results in a simple shift of the spectral parameter, $u \rightarrow u-v_1$ in the first operator Q_- and $u \rightarrow u-v_2$ in the second one. Eliminating the redundant shift of the spectral parameter ($u-v \rightarrow u$) we have

$$\frac{1}{1-q} \cdot T_s(u|q) = Q_+(u-s) Q_-(u+s+1) = Q_-(u+s+1) Q_+(u-s). \quad (4.10)$$

We derive this from the underlying local factorization relations for building blocks of these general transfer matrices. Starting from (4.5) and choosing the first space to be the local quantum space $\mathbb{U}_{-\ell}$ in site k , the second space to be the auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_0]$ and the third space to be another copy of the auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_{0'}]$

$$\mathbb{R}_{00'}^1(v_1|w_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) =$$

$$= \mathbb{R}_{k0}(u_1, u_2|v_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, v_2) \mathbb{R}_{00'}^1(v_1|w_1, w_2) \quad (4.11)$$

We have to do the appropriate limiting procedure in the previous relation taking at first $w_1 \rightarrow \infty$ and then $w_2 \rightarrow \infty$ which leads to

$$\begin{aligned} & P_{00'}(-)^{z_{0'}\partial_{0'}} e^{z_0\partial_{0'}} \cdot e^{z_{0'}\partial_k} \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}^-(u_1, u_2|v_1) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|v_2) \cdot P_{00'}(-)^{z_{0'}\partial_{0'}} e^{z_0\partial_{0'}} \end{aligned} \quad (4.12)$$

We present the derivation of (4.12) in Appendix B. This local intertwining relation leads in the standard way to the relation for the corresponding transfer matrices

$$\begin{aligned} & \text{tr}_{0'} \left[q^{z_{0'}\partial_{0'}} e^{z_{0'}\partial_1} \dots e^{z_{0'}\partial_N} \right] \cdot \text{tr}_0 \left[q^{z_0\partial_0} \mathbb{R}_{10}(u_1, u_2|v_1, v_2) \dots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2) \right] = \\ & = \text{tr}_0 \left[q^{z_0\partial_0} \mathbb{R}_{10}^-(u_1, u_2|v_1) \dots \mathbb{R}_{N0}^-(u_1, u_2|v_1) \right] \cdot \text{tr}_{0'} \left[q^{z_{0'}\partial_{0'}} \mathbb{R}_{10'}^+(u_1, u_2|v_2) \dots \mathbb{R}_{N0'}^+(u_1, u_2|v_2) \right]. \end{aligned}$$

As one can easily see the trace $\text{tr}_{0'} [e^{z_{0'}\partial_1} \dots e^{z_{0'}\partial_N}]$ does not converge so that one needs a regularization. The regularized expressions contains poles at $q \rightarrow 1$. Indeed, we readily compute the trace

$$\text{tr}_{0'} \left[q^{z_{0'}\partial_{0'}} e^{z_{0'}\partial_1} \dots e^{z_{0'}\partial_N} \right] = \text{tr}_{0'} q^{z_{0'}\partial_{0'}} = \frac{1}{1-q}$$

and we obtain the second equality in (4.10). We see that the regularization violating sl_2 symmetry is unavoidable in the present construction.

Starting from (4.6) instead leads to the first factorization relation in (4.10) by analogous steps. The relation (4.10) also states the commutativity of transfer matrices constructed from \mathbb{R}^+ and \mathbb{R}^- .

Above we have formulated local factorization relations (3.17) and (3.19) connecting \mathbb{R}^- with \mathbb{R}^1 and \mathbb{R}^+ with \mathbb{R}^2 :

$$\begin{aligned} \mathbb{R}_{12}^-(u_1, u_2|v_1) &= P_{12} \mathbb{R}_{12}^1(u_1|v_1, u_2) r_{12}^-(u_1|u_2) \\ \mathbb{R}_{12}^+(u_1, u_2|v_2) &= P_{12} r_{12}^+(u_1|u_2) \mathbb{R}_{12}^2(u_1, u_2|v_2) \end{aligned}$$

Analogous relations hold on the level of transfer matrices,

$$P q^{z_1\partial_1} \cdot Q_-(u) = Q_1(u|q) \cdot q_- \quad (4.13)$$

and

$$P q^{z_1\partial_1} \cdot Q_+(u) = q_+ \cdot Q_2(u|q), \quad (4.14)$$

where operators the q_+ and q_- are auxiliary transfer matrices constructed from r_{k0}^+ and r_{k0}^-

$$q_+ = \text{tr}_0 \left[q^{z_0\partial_0} r_{10}^+(u_1|u_2) \dots r_{N0}^+(u_1|u_2) \right], \quad (4.15)$$

$$q_- = \text{tr}_0 \left[q^{z_0\partial_0} r_{10}^-(u_1|u_2) \dots r_{N0}^-(u_1|u_2) \right]. \quad (4.16)$$

They are almost inverse to each other because choosing the parameters in (4.10) in a special way we obtain

$$\frac{1}{1-q} \cdot P q^{z_1\partial_1} = q_- \cdot q_+ = q_+ \cdot q_- \quad (4.17)$$

The operators Q_1 and Q_2 are q -regularized Baxter operators constructed in part I

$$Q_1(u - v_1|q) = \text{tr}_0 \left[q^{z_0\partial_0} \mathbb{R}_{10}^1(u_1|v_1, u_2) \dots \mathbb{R}_{N0}^1(u_1|v_1, u_2) \right], \quad (4.18)$$

$$Q_2(u - v_2|q) = \text{tr}_0 \left[q^{z_0\partial_0} \mathbb{R}_{10}^2(u_1, u_2|v_2) \dots \mathbb{R}_{N0}^2(u_1, u_2|v_2) \right], \quad (4.19)$$

which obey the factorization relation

$$P q^{z_1 \partial_1} \cdot T_s(u|q) = Q_2(u-s|q) Q_1(u+s+1|q) = Q_1(u+s+1|q) Q_2(u-s|q) \quad (4.20)$$

and commute among themselves. In Appendix B we list the local factorization relations which produce (4.13) by implementing the reduction in (4.11).

Thus we have derived all factorization properties of transfer matrices from underlying local relations which are certain reductions of (4.5) and (4.6). With respect to the commutativity properties it holds that q -regularized transfer matrices constructed from local blocks $\mathbb{R}, \mathbb{R}^\pm, \mathbb{r}^\pm, \mathbb{R}^1, \mathbb{R}^2$ and P_{k0} commute among themselves. We deduce these properties from the underlying local relations which now are reductions of the general Yang-Baxter relation (4). The corresponding calculations can be found in Appendix B.

The transfer matrices Q_+ and Q_- constructed from operators \mathbb{R}^+ and \mathbb{R}^- have all properties of the Q -operators, introduced by R.Baxter [5,6]. All commutativity properties have been proven already

$$\begin{aligned} [T_s(u|q), Q_+(v)] &= [T_s(u|q), Q_-(v)] = [Q_+(u), Q_-(v)] = 0 \\ [Q_+(u), Q_+(v)] &= [Q_-(u), Q_-(v)] = 0 \end{aligned} \quad (4.21)$$

so that we shall focus in the following on the Baxter equation for Q_\pm .

In part I we have shown how to prove the Baxter equations starting from RLL-relation (3.8) and (3.9) (restriction to invariant two dimensional subspace of the tree-term relations (4.5) and (4.6)). One can accomplish the analogous construction for the present pair of Baxter operators using the RLL-relations (3.16) and (3.18).

But it is much simpler to use the connection between the operators Q_+, Q_- and the operators Q_1, Q_2 expressed by the factorization relations (4.13) and (4.14). We have already mentioned above that all families of transfer matrices $Q_{1,2}(u|q), Q_\pm(u), q_\pm, P q^{z_1 \partial_1}$ commute among themselves.

The operators $Q_{1,2}(u|q)$ respect the following Baxter equation (compare part I)²

$$t(u|q) Q_1(u|q) = Q_1(u+1|q) + q \cdot (u_1 u_2)^N \cdot Q_1(u-1|q) \quad (4.22)$$

$$t(u|q) Q_2(u|q) = q \cdot Q_2(u+1|q) + (u_1 u_2)^N \cdot Q_2(u-1|q) \quad (4.23)$$

where

$$t(u|q) = \text{tr} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} L_1(u) L_2(u) \cdots L_N(u). \quad (4.24)$$

Then using the relation between the two sets of Baxter operators and the commutativity of all transfer matrices we obtain immediately the desired Baxter equations

$$t(u|q) Q_-(u) = Q_-(u+1) + q \cdot (u_1 u_2)^N \cdot Q_-(u-1), \quad (4.25)$$

$$t(u|q) Q_+(u) = q \cdot Q_+(u+1) + (u_1 u_2)^N \cdot Q_+(u-1), \quad (4.26)$$

since neither q_\pm nor $P q^{z_1 \partial_1}$ depend on the spectral parameter.

² In order to obtain Baxter relation in the standard form we change normalization of R -operators $R(u_1, u_2|v_1, v_2) \rightarrow (-1)^{u_1-v_1} R(u_1, u_2|v_1, v_2)$, $R^-(u_1, u_2|v_1) \rightarrow (-1)^{u_1-v_1} R^-(u_1, u_2|v_1)$ and $R^1(u_1|v_1, v_2) \rightarrow (-1)^{u_1-v_1} R^1(u_1|v_1, v_2)$. In the other parts of the paper we do not retain this normalization factor since it can be restored easily.

4.2 Explicit action on polynomials

The previous construction of Baxter operators is a mostly algebraic one. Now we are going to derive explicit formulae which show how the constructed operators act on polynomials. At first we calculate the traces in the definitions of the simple operators q_+ and q_- . Then we take into account the explicit formulae for $Q_{1,2}(u|q)$ obtained in part I and finally combine them by means of the factorization relations (4.13), (4.14) .

We transform (4.16), (4.15) using (3.15) and (3.13)

$$q_- = \text{tr}_0 \left[q^{z_0 \partial_0} P_{10} e^{-z_0 \partial_1} \dots P_{N0} e^{-z_0 \partial_N} \right] \cdot \Pi_- \quad (4.27)$$

$$q_+ = \Pi_+ \cdot \text{tr}_0 \left[q^{z_0 \partial_0} P_{10} e^{z_1 \partial_0} \dots P_{N0} e^{z_N \partial_0} \right] \quad (4.28)$$

where we suppress for brevity the dependence on ℓ in our notations and extract the operators

$$\Pi_- \equiv \prod_{k=1}^N \frac{(-)^{z_k \partial_k}}{\Gamma(z_k \partial_k - 2\ell)} \quad , \quad \Pi_+ \equiv \prod_{k=1}^N \Gamma(z_k \partial_k - 2\ell) .$$

These operators are relatives of the projectors on the finite-dimensional subspace that appear in the case of half-integer ℓ used in part I. The trace in (4.27) can be calculated explicitly (compare part I, section 4.3 and Appendix B)

$$\text{tr}_0 \left[q^{z_0 \partial_0} P_{10} e^{-z_0 \partial_1} \dots P_{N0} e^{-z_0 \partial_N} \right] = P q^{z_1 \partial_1} e^{-z_2 \partial_1} e^{-z_3 \partial_2} \dots e^{-z_0 \partial_N} \Big|_{z_0 = \frac{z_1}{q}} ,$$

and this operator acts on polynomials in the following way

$$\begin{aligned} \text{tr}_0 \left[q^{z_0 \partial_0} P_{10} e^{-z_0 \partial_1} \dots P_{N0} e^{-z_0 \partial_N} \right] \cdot \Psi(z_1, \dots, z_N) &= \\ &= \Psi(qz_N - z_1, z_1 - z_2, \dots, z_{N-1} - z_N) . \end{aligned} \quad (4.29)$$

In order to calculate the trace in (4.28) we act on the function $z_0^k \Psi(z_1, \dots, z_N)$

$$\begin{aligned} q^{z_0 \partial_0} P_{10} e^{z_1 \partial_0} \dots P_{N0} e^{z_N \partial_0} \cdot z_0^k \Psi(z_1, \dots, z_N) &= \\ &= (qz_0 + z_1 + \dots + z_N)^k \Psi(qz_0, qz_0 + z_1, qz_0 + z_1 + z_2, \dots, qz_0 + z_1 + \dots + z_N) \end{aligned}$$

and then apply the formula $\sum_{k=0}^{\infty} \frac{1}{k!} \partial_0^k \cdot (a + b \cdot z_0)^k \Phi(z_0) \Big|_{z_0=0} = \frac{1}{1-b} \cdot \Phi\left(\frac{a}{1-b}\right)$ proven in part I (Appendix B), so that one obtains

$$\begin{aligned} \text{tr}_0 \left[q^{z_0 \partial_0} P_{10} e^{z_1 \partial_0} \dots P_{N0} e^{z_N \partial_0} \right] \Psi(z_1, \dots, z_N) &= \\ &= \frac{1}{1-q} \cdot \Psi(z_0, z_0 + z_1, z_0 + z_1 + z_2, \dots, z_0 + z_1 + \dots + z_{N-1}) \Big|_{z_0 \rightarrow \frac{q}{1-q} \cdot (z_1 + \dots + z_N)} . \end{aligned} \quad (4.30)$$

Let us recall the formulae for $Q_{1,2}(u|q)$ from part I. The action of the operator $Q_1(u|q)$ on polynomials is

$$Q_1(u|q) \Psi(\vec{z}) = R_1(\lambda_1 \partial_{\lambda_1}) \dots R_1(\lambda_N \partial_{\lambda_N}) \Big|_{\lambda=1} \cdot \frac{1}{1 - q \bar{\lambda}_1 \dots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}) , \quad (4.31)$$

where

$$\Lambda'_q = \begin{pmatrix} 1 - \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & 1 - \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_3} & \frac{1}{\lambda_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 - \frac{1}{\lambda_{N-1}} & \frac{1}{\lambda_{N-1}} \\ \frac{1}{q\lambda_N} & 0 & 0 & 0 & \dots & 1 - \frac{1}{\lambda_N} \end{pmatrix}, \quad R_1(x) \equiv \frac{\Gamma(x - 2\ell)}{\Gamma(x + 1 - \ell - u)}$$

In contrast to Q_1 the renormalized operator Q_2 possess a more elegant representation

$$Q(u|q) = \frac{\Gamma^N(-2\ell)}{\Gamma^N(-\ell + u)} \cdot Q_2(u|q). \quad (4.32)$$

Indeed, its action on the generating function of the representation in quantum space looks very simple

$$\begin{aligned} Q(u|q) : (1 - x_1 z_1)^{2\ell} \dots (1 - x_N z_N)^{2\ell} &\mapsto \\ \mapsto (1 - q x_1 z_N)^{\ell - u} (1 - x_1 z_1)^{\ell + u} \dots (1 - x_N z_{N-1})^{\ell - u} (1 - x_N z_N)^{\ell + u}. \end{aligned} \quad (4.33)$$

Now we combine the factorization relation (4.13), the expression for q_+ (4.28), (4.30), expression for $Q(u|q)$ (4.33) and introduce the renormalized operator

$$Q^+(u) = \frac{1 - q}{\Gamma^N(-\ell + u)} \cdot Q_+(u) \quad (4.34)$$

which maps polynomials in $z_1 \dots z_N$ to polynomials in $u, z_1 \dots z_N$

$$Q^+(u) : \mathbb{C}[z_1 \dots z_N] \rightarrow \mathbb{C}[u, z_1 \dots z_N]$$

We obtain that the renormalized Baxter operator acts on the generating function as follows

$$\begin{aligned} Q^+(u) : \prod_{k=1}^N (1 - x_k z_k)^{2\ell} &\mapsto \prod_{k=1}^N \frac{\Gamma(z_k \partial_k - 2\ell)}{\Gamma(-2\ell)} \times \\ &\times \prod_{k=1}^N \left(1 - \frac{x_k}{1 - q} \cdot (z_{1,k-1} + q z_{k,N}) \right)^{\ell - u} \left(1 - \frac{x_k}{1 - q} \cdot (z_{1,k} + q z_{k+1,N}) \right)^{\ell + u}, \end{aligned} \quad (4.35)$$

where

$$z_{1,k} \equiv z_1 + z_2 + \dots + z_k \quad ; \quad z_{k,N} \equiv z_k + z_{k+1} + \dots + z_N.$$

$Q^+(u)$ is normalized in such a way that $Q^+(u) : 1 \mapsto 1$.

Combining the factorization relation (4.13), the expression for q_- (4.27), (4.29) and the expression for $Q_1(u|q)$ (4.31) we can calculate explicitly how Q_- acts on polynomials.

5 Finite-dimensional representations

In the previous section we have assumed the spin parameter ℓ to be a generic complex number and consequently the quantum space of the model was infinite-dimensional. Under this assumptions we have concentrated on the algebraic properties of sl_2 -invariant transfer matrices constructed from the operators \mathbb{R}^+ and \mathbb{R}^- and have demonstrated the key properties that allows to call them Q -operators: commutativity and Baxter equation. Now we are going to consider the special situation of integer or half-integer spin ℓ . In this case the infinite-dimensional representation becomes

reducible and there appears an invariant subspace which is the standard finite-dimensional irreducible representation labeled by (half)-integer spin ℓ .

It turns out that the previous construction for infinite-dimensional representations can be transferred straightforwardly to the finite-dimensional case. We only have to substitute integer or half-integer values of ℓ in the formulae obtained above and restrict the operators to the irreducible finite-dimensional subspace. In doing so we do not face any obstacles. In other words, the Baxter operators Q_{\pm} and the general transfer matrix T constructed above do not map beyond the detached finite-dimensional subspace at (half)-integer ℓ .

Our presentation will be the following:

- We shall work with the restriction of the general \mathbb{R} -operator and the operators \mathbb{R}^+ , \mathbb{R}^- to the invariant subspace appearing for half-integer or integer values of the spin.
- We use these restricted operators \mathbb{R} , \mathbb{R}^+ , \mathbb{R}^- as building blocks for the construction of the corresponding transfer matrices.
- Using local relations we prove the factorization of the general transfer matrix into the product of the corresponding Q -operators.

To start with we consider simple examples of restriction to invariant subspace in the next subsection.

5.1 Examples of restriction: two-dimensional representations

Consider the general \mathbb{R} -operator (3.11). It acts in the space $\mathbb{U}_{-\ell} \otimes \mathbb{U}_{-s}$ and has the form

$$\mathbb{R}_{12}(u|\ell, s) = P_{12} \cdot \frac{\Gamma(z_{21}\partial_2 - 2\ell)}{\Gamma(z_{21}\partial_2 - \ell - s - u)} \cdot \frac{\Gamma(z_{12}\partial_1 - \ell - s + u)}{\Gamma(z_{12}\partial_1 - 2\ell)} \quad (5.1)$$

after passing to the spin parameter notation (4.2)

$$u_1 = u - \ell - 1, \quad u_2 = u + \ell; \quad v_1 = v - s - 1, \quad v_2 = v + s. \quad (5.2)$$

In discrete points $\ell = \frac{n}{2}, n = 0, 1, 2, 3, \dots$ the module $\mathbb{U}_{-\ell}$ becomes reducible. Therefore it is possible to restrict the \mathbb{R} -operator to the space $\mathbb{V}_n \otimes \mathbb{U}_{-s}$. In the case $\ell = \frac{1}{2}$ we have

$$\mathbb{V}_1 \otimes \mathbb{U}_{-s} \sim \mathbb{C}^2 \otimes \mathbb{U}_{-s}$$

and the restricted \mathbb{R} -operator acts on functions of the form

$$\Psi(z_1, z_2) = \phi(z_2) + z_1 \psi(z_2). \quad (5.3)$$

In part I (section 5.1) we have performed the detailed calculations of such a restriction analyzing the limit $\varepsilon \rightarrow 0$, $2\ell = n - \varepsilon$ and obtained that in the basis $\mathbf{e}_1 = -z_1, \mathbf{e}_2 = 1$ restricted \mathbb{R} -operator has the form

$$\mathbb{R}(u|\frac{1}{2}, s)|_{\mathbb{V}_1} = -\frac{\Gamma(-s - \frac{1}{2} + u)}{\Gamma(-s + \frac{1}{2} - u)} \cdot L(u + \frac{1}{2}|s). \quad (5.4)$$

One can easily perform similar calculations for the operators \mathbb{R}^+ and \mathbb{R}^- (3.17), (3.19). In the spin parameter notations they have the form

$$\mathbb{R}_{12}^+(u|\ell, s) = P_{12} \cdot \Gamma(z_2\partial_2 - 2\ell) e^{z_1\partial_2} \cdot \frac{\Gamma(z_{12}\partial_1 - \ell - s + u)}{\Gamma(z_{12}\partial_1 - 2\ell)}, \quad (5.5)$$

$$\mathbb{R}_{12}^-(u|\ell, s) = P_{12} \cdot \frac{\Gamma(z_{21}\partial_2 - 2\ell)}{\Gamma(z_{21}\partial_2 - \ell - s - u)} \cdot e^{-z_2\partial_1} \frac{(-)^{z_1\partial_1}}{\Gamma(z_1\partial_1 - 2\ell)}. \quad (5.6)$$

Using the notations (2.6), (2.7) we have

$$\mathbb{R}^+(u|\frac{1}{2}, s)|_{\mathbb{V}^1} = \Gamma(-s - \frac{1}{2} + u) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot L^+(u - s - \frac{1}{2}), \quad (5.7)$$

$$\mathbb{R}^-(u|\frac{1}{2}, s)|_{\mathbb{V}^1} = \Gamma^{-1}(-s + \frac{1}{2} - u) \cdot L^-(u + s + \frac{1}{2}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.8)$$

This example explicitly shows that contrary to the operator \mathbb{R}^2 studied in part I, the operators \mathbb{R}^+ , \mathbb{R}^- share the important property with the general \mathbb{R} -operator – they all preserve the invariant two-dimensional subspace for half-integer spin. As a consequence the properties of the general transfer matrix T as well as the operators Q_- , Q_+ are also similar with respect to the restriction to the finite-dimensional subspace.

Now we are going to consider the restriction of the simple operator $e^{-z_1\partial_2}$. It acts on the space of functions (5.3)

$$e^{-z_1\partial_2} [\phi(z_2) + z_1\psi(z_2)] = \phi(z_2) - z_2\psi(z_2) + z_1\psi(z_2)$$

In the basis form we have

$$\begin{aligned} e^{-z_1\partial_2} \mathbf{e}_1 &= \mathbf{e}_1 \cdot 1 + \mathbf{e}_2 \cdot z_2 \\ e^{-z_1\partial_2} \mathbf{e}_2 &= \mathbf{e}_2 \cdot 1 \end{aligned}$$

and in matrix form

$$e^{-z_1\partial_2}|_{\mathbb{V}^1} = \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix}. \quad (5.9)$$

Let us now turn to the local relation (4.12) which produces the factorization relation for the general transfer matrix and transform it to the form (indices 1, 2 denote two auxiliary spaces and index 0 denotes the quantum space in one site of the chain)

$$e^{z_2\partial_1} \cdot \mathbb{R}_{01}^-(u_1, u_2|v_1) \cdot \mathbb{R}_{02}^+(u_1, u_2|v_2) = e^{-z_1\partial_0} \cdot \mathbb{R}_{02}(u_1, u_2|v_1, v_2) \cdot e^{z_2\partial_1} \quad (5.10)$$

Then we perform the restriction of this relation to the two-dimensional subspace in quantum space at $\ell = \frac{1}{2}$. We take into account (5.5), (5.6), (5.9) and obtain (2.9)

$$e^{z_2\partial_1} \cdot L_1^-(u_2) L_2^+(u_1) = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} L_2(u_1, u_2) \cdot e^{z_2\partial_1} \quad (5.11)$$

which underlies the construction of [9].

Similarly one can rewrite (B.3) in the form

$$\begin{aligned} & r_{12}^+(v_1|v_2) \cdot \mathbb{R}_{01}^+(u_1, u_2|v_2) \cdot \mathbb{R}_{02}^-(u_1, u_2|v_1) = \\ & = \Gamma(z_0\partial_0 + u_1 - u_2 + 1) e^{z_0\partial_1} \Gamma^{-1}(z_0\partial_0 + u_1 - u_2 + 1) \cdot (-)^{z_2\partial_2} \mathbb{R}_{02}(u_1, u_2|v_1, v_2) (-)^{z_2\partial_2} \cdot r_{12}^+(v_1|v_2) \end{aligned}$$

and perform the restriction

$$\Gamma(z_0\partial_0 - 2\ell) e^{z_0\partial_1} \Gamma^{-1}(z_0\partial_0 - 2\ell) \Big|_{\mathbb{V}^1, \ell \rightarrow \frac{1}{2}} = \begin{pmatrix} 1 & \partial_1 \\ 0 & 1 \end{pmatrix}.$$

In this way one obtains the second local factorization relation (2.10).

The approach proposed here reproduces the local formulae (2.9), (2.10) and in some sense explains the origin of these peculiar relations on which the construction in [9] relies: finally everything is based on the three-term relations (4.5), (4.6). On the other hand our construction in part I is solely based on (4.5), (4.6). This means, that both constructions are equivalent generally, because they can be founded on the same basis. However each of them has specific properties.

5.2 Finite-dimensional operators \mathbf{R} , \mathbf{R}^+ , \mathbf{R}^-

In the previous subsection we have demonstrated the restriction of operators \mathbb{R} , \mathbb{R}^+ , \mathbb{R}^- acting in the space $\mathbb{U}_{-\ell} \otimes \mathbb{U}_{-s}$ to invariant subspace $\mathbb{V}_n \otimes \mathbb{U}_{-s}$ in the case $n = 1$. For integer values $2\ell = n$ the module $\mathbb{U}_{-\ell}$ becomes reducible and the action of these three operators on the first factor space becomes reducible. This can be checked by analyzing the limits $\varepsilon \rightarrow 0$, $2\ell = n - \varepsilon$ in the products $\mathbf{R} = \mathbf{R}^1 \mathbf{R}^2$ (3.11), $\mathbf{R}^+ = \mathbf{r}^+ \mathbf{R}^2$ (3.17), $\mathbf{R}^- = \mathbf{R}^1 \mathbf{r}^-$ (3.19).

In the following it is convenient to use the projection operators

$$\Pi_i^n z_i^k = z_i^k, \quad k \leq n \quad ; \quad \Pi_i^n z_i^k = 0, \quad k > n. \quad (5.12)$$

Thus we restrict the operators as follows

$$\mathbf{R}_{12}(u_1, u_2 | v_1, v_2) = \lim_{\varepsilon \rightarrow 0} \mathbb{R}_{12}(u_1 + \frac{\varepsilon}{2}, u_2 - \frac{\varepsilon}{2} | v_1, v_2) \Pi_1^n \quad (5.13)$$

$$\mathbf{R}_{12}^+(u_1, u_2 | v_2) = \lim_{\varepsilon \rightarrow 0} \mathbb{R}_{12}^+(u_1 + \frac{\varepsilon}{2}, u_2 - \frac{\varepsilon}{2} | v_2) \Pi_1^n \quad (5.14)$$

$$\mathbf{R}_{12}^-(u_1, u_2 | v_1) = \lim_{\varepsilon \rightarrow 0} \mathbb{R}_{12}^-(u_1 + \frac{\varepsilon}{2}, u_2 - \frac{\varepsilon}{2} | v_1) \Pi_1^n \quad (5.15)$$

where the parameters are

$$u_1 = u - \frac{n}{2} - 1, \quad u_2 = u + \frac{n}{2} \quad ; \quad v_1 = v - s - 1, \quad v_2 = v + s \quad (5.16)$$

We shall write in boldface style operators in integer or half-integer spin cases restricted to the finite-dimensional irreducible subspace. Let us mention that \mathbf{R} , \mathbf{R}^+ , \mathbf{R}^- do not map beyond subspace $\mathbb{V}_n \otimes \mathbb{U}_{-s}$ as it should be.

In Appendix C we calculate such restrictions to the space $\mathbb{V}_n \otimes \mathbb{U}_{-s}$ of the \mathbb{R} -operator in general case, i.e. at $2\ell \rightarrow n$, $n = 0, 1, 2, \dots$. We also specify there the explicit expression for \mathbf{R}^+ , the restriction of \mathbb{R}^+ , and \mathbf{R}^- , the restriction of \mathbb{R}^- . We present them as certain finite-dimensional differential operators since such a form is convenient. Then choosing particular values of n one can construct the corresponding matrices as we have done in the previous section.

In the Section 3.2.2 performing reductions in RLL-relations we have introduced the operators \mathbf{R}^+ and \mathbf{R}^- as certain limits of the general \mathbf{R} -operator (3.17), (3.19). One can perform similar reduction of the operator \mathbf{R} cancelling the leading dependence on the large v_1 or v_2 like in (3.17), (3.19). In this way we obtain³ once more the operators \mathbf{R}^+ and \mathbf{R}^- ,

$$v_1^{z_1 \partial_1} \mathbf{R}(u_1, u_2 | v_1, v_2) \rightarrow \mathbf{R}^+(u_1, u_2 | v_2) \quad \text{at} \quad v_1 \rightarrow \infty, \quad (5.17)$$

$$\mathbf{R}(u_1, u_2 | v_1, v_2) v_2^{-z_1 \partial_1} \rightarrow \mathbf{R}^-(u_1, u_2 | v_1) \quad \text{at} \quad v_2 \rightarrow \infty. \quad (5.18)$$

This shows that we can obtain \mathbf{R}^+ and \mathbf{R}^- in two ways: at first perform the reduction of the general \mathbb{R} -operator taking one of its parameters to infinity and then restrict to the finite-dimensional subspace or inversely at first restrict to the subspace and only then perform the reduction taking the appropriate limit. (5.17), (5.18) can be obtained using Stirling's formula (A.3) and performing calculation analogous to the one in Appendix A.

³ The relations (5.17) and (5.18) should be understood up to normalization which we fix as in (C.4) and (C.5)

5.3 The general transfer matrices and \mathbf{Q}_\pm -operators

After the necessary preparation in the previous section we are going to the construction of the general transfer matrix (4.1) out of the new building blocks \mathbf{R}_{k0} if ℓ is (half)-integer

$$\mathbf{T}_s(u) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}(u|\frac{n}{2}, s) \mathbf{R}_{20}(u|\frac{n}{2}, s) \cdots \mathbf{R}_{N0}(u|\frac{n}{2}, s) \quad (5.19)$$

which is well defined on the finite-dimensional quantum space of the model. In this case the trace over infinite-dimensional auxiliary space $\mathbb{C}[z_0]$ diverges without q -regularization.

Now we consider the factorization of the general transfer matrix (1.1). We start as before with the three term relation (4.5) and restrict it on site k to $\mathbb{V}^n \otimes \mathbb{C}[z_0] \otimes \mathbb{C}[z_{0'}]$ at $\ell = \frac{n}{2}$

$$\begin{aligned} \mathbb{R}_{00'}^1(v_1|w_1, w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ = \mathbf{R}_{k0}(u_1, u_2|v_1, w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, v_2) \mathbb{R}_{00'}^1(v_1|w_1, w_2) \end{aligned} \quad (5.20)$$

Then we have to do the appropriate limiting procedure in the previous relation taking at first $w_1 \rightarrow \infty$ and then $w_2 \rightarrow \infty$. This leads to

$$\begin{aligned} \mathbf{P}_{00'}(-)^{z_{0'} \partial_{0'}} e^{z_0 \partial_0} \cdot e^{z_{0'} \partial_k} \Pi_k^n \cdot \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ = \mathbf{R}_{k0}^-(u_1, u_2|v_1) \cdot \mathbf{R}_{k0'}^+(u_1, u_2|v_2) \cdot \mathbf{P}_{00'}(-)^{z_{0'} \partial_{0'}} e^{z_0 \partial_0} \end{aligned} \quad (5.21)$$

We present the derivation of (5.21) in Appendix C. This local relation leads in the standard way to the factorization relation for corresponding regularized transfer matrices

$$\frac{1}{1-q} \cdot \mathbf{T}_s(u) = \mathbf{Q}_+(u-s) \mathbf{Q}_-(u+s+1) = \mathbf{Q}_-(u+s+1) \mathbf{Q}_+(u-s) \quad (5.22)$$

where \mathbf{Q}_- , \mathbf{Q}_+ are transfer matrices constructed from building blocks \mathbf{R}_{k0}^- , \mathbf{R}_{k0}^+

$$\mathbf{Q}_-(u-v_1) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^-(u_1, u_2|v_1) \cdots \mathbf{R}_{N0}^-(u_1, u_2|v_1), \quad (5.23)$$

$$\mathbf{Q}_+(u-v_2) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^+(u_1, u_2|v_2) \cdots \mathbf{R}_{N0}^+(u_1, u_2|v_2) \quad (5.24)$$

The second variant of the factorization in the above relation can be obtained similarly from appropriate local relation.

This construction is analogous to the one in Section 4, where ℓ was a generic complex number. The proof of the commutativity

$$[\mathbf{T}_s(u), \mathbf{Q}_\pm(v)] = 0 \quad ; \quad [\mathbf{Q}_\pm(u), \mathbf{Q}_\pm(v)] = 0 \quad ; \quad [\mathbf{Q}_-(u), \mathbf{Q}_+(u)] = 0 \quad (5.25)$$

uses the general Yang-Baxter equation and also goes parallel the corresponding derivation given in Section 4.

Let us stress once more that we have based each relation for the general transfer matrix and Baxter operators on a corresponding local relation for their building blocks. We derive such local relations from three basic relations (4), (4.5), (4.6) performing appropriate limits. The general Yang-Baxter relation (4) produces local relations implying commutativity of the corresponding transfer matrices and (4.5), (4.6) produce local relations implying factorization relations for transfer matrices. Considering the construction with finite-dimensional representations in the quantum space we at first restrict the basic relations (4), (4.5), (4.6) to the finite-dimensional subspace in quantum space and only then perform appropriate limiting procedures which exclude a part of the parameters. In this way, for example, we obtain the sequence of relations (4.5) \rightarrow (5.20) \rightarrow (5.21).

We have shown that the local operators \mathbb{R}^+ and \mathbb{R}^- admit the restriction to the finite-dimensional subspace in quantum space (5.14), (5.15) if ℓ is (half)-integer. Consequently the same is valid for the Baxter operators constructed out of them. Indeed, we have in the limit $\varepsilon \rightarrow 0$, $2\ell = n - \varepsilon$

$$\mathbf{Q}_{\pm}(u) = \lim_{\varepsilon \rightarrow 0} \mathbf{Q}_{\pm}(u)|_{\ell=\frac{n-\varepsilon}{2}} \cdot \Pi^n, \quad \text{where} \quad \Pi^n \equiv \Pi_1^n \Pi_2^n \cdots \Pi_N^n. \quad (5.26)$$

Before discussing the Baxter equation we derive some formulae connecting Baxter operators \mathbf{Q}_{\pm} introduced above with Baxter operators $\mathbf{Q}_{1,2}$ constructed in part I. This connection implies the form of the Baxter equations for operators \mathbf{Q}_{\pm} .

5.4 The operators \mathbf{R}^1 , \mathbf{R}^2 , \mathbf{S}

Along with the operator $\mathbf{R}(u_1, u_2|v_1, v_2)$ we have also introduced in part I the operators $\mathbf{R}^1(u_1|v_1, v_2)$ and $\mathbf{R}^2(u_1, u_2|v_2)$ where parameters u_1, u_2, v_1, v_2 respect the relation (5.16), i.e. the quantum spin ℓ is (half)-integer. They are obtained from the former one by imposing additional relations on parameters. Taking the limit $\delta \rightarrow 0$ we obtain

$$\mathbf{R}_{12}(u_1, u_2|v_1, u_2 - \delta) \rightarrow \delta^{-1} \cdot \mathbf{R}_{12}^1(u_1|v_1, u_2), \quad (5.27)$$

$$\mathbf{R}_{12}(u_1, u_2|u_1 + \delta, v_2) \rightarrow \delta \cdot \mathbf{R}_{12}^2(u_1, u_2|v_2), \quad (5.28)$$

$$\mathbf{R}_{12}(u_1, u_2|u_1 + \delta, u_2 - \delta) \rightarrow \mathbf{S}_{12} \equiv \mathbf{P}_{12} \cdot e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \cdot \Pi_1^n \quad (5.29)$$

The operators \mathbf{R}^1 and \mathbf{R}^2 are finite-dimensional analogues of \mathbb{R}^1 and \mathbb{R}^2 (3.10). They do not map beyond $\mathbb{V}_n \otimes \mathbb{U}_{-s}$. Indeed they are special limits of \mathbf{R} which is well defined on the finite-dimensional subspace. Let us mention that the operator \mathbb{R}^2 does not admit a restriction to the finite-dimensional subspace, i.e. it maps beyond $\mathbb{V}_n \otimes \mathbb{U}_{-s}$. It is the main reason for constructing building the blocks for transfer matrices from \mathbf{R} instead of taking them to be \mathbb{R}^1 or \mathbb{R}^2 .

The operator \mathbf{S}_{12} is a nontrivial analogue of the transposition operator \mathbf{P}_{12} for the finite-dimensional construction. It also does not map beyond $\mathbb{V}_n \otimes \mathbb{U}_{-s}$. Let us recall that $\mathbb{R}_{12}(u_1, u_2|u_1, u_2) = \mathbf{P}_{12}$ in contrast to (5.29). This example clearly shows that the order in which we impose the relations on parameters and take the spin in quantum space ℓ to be (half)-integer is important.

In the Section 3.2.1 we have considered the operators \mathbf{r}^+ and \mathbf{r}^- which are certain reductions of \mathbf{R}^1 and \mathbf{R}^2 . Now we are going to introduce their analogues for the finite-dimensional case, i.e. we perform the reduction of operators \mathbf{R}^1 and \mathbf{R}^2 . We adopt here the same rule (3.15), (3.13) to perform the limiting procedure,

$$v_1^{z_1 \partial_1} \mathbf{R}^1(u_1|v_1, v_2) \rightarrow \mathbf{r}^+(u_1|v_2) \quad \text{at} \quad v_1 \rightarrow \infty, \quad (5.30)$$

$$\mathbf{R}^2(u_1, u_2|v_2) v_2^{-z_1 \partial_1} \rightarrow \mathbf{r}^-(u_1|u_2) \quad \text{at} \quad v_2 \rightarrow \infty. \quad (5.31)$$

Explicit expression for the operators \mathbf{R}^1 , \mathbf{R}^2 , \mathbf{r}^- and \mathbf{r}^+ can be found in Appendix C.

5.5 Relations between \mathbf{Q}_{\pm} and $\mathbf{Q}_{1,2}$

In Section 4 we have pointed out the connection between two constructions of Baxter operators for infinite-dimensional representations. Now we are going to do the same for the finite-dimensional case. In part I (compare Appendix C there) we have introduced the q -regularization into the traces for Baxter operators \mathbf{Q}_1 and \mathbf{Q}_2 .

$$\mathbf{Q}_1(u - v_1) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^1(u_1|v_1, u_2) \cdots \mathbf{R}_{N0}^1(u_1|v_1, u_2),$$

$$\begin{aligned}\mathbf{Q}_2(u - v_2) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^2(u_1, u_2 | v_2) \cdots \mathbf{R}_{N0}^2(u_1, u_2 | v_2) \\ \mathbf{S} &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{S}_{10} \mathbf{S}_{20} \cdots \mathbf{S}_{N0},\end{aligned}\tag{5.32}$$

They obey the factorization relation

$$\mathbf{S} \mathbf{T}_s(u) = \mathbf{Q}_2(u - s) \mathbf{Q}_1(u + s + 1) = \mathbf{Q}_1(u + s + 1) \mathbf{Q}_2(u - s)\tag{5.33}$$

very similar to (5.22) and satisfy the Baxter equations

$$\begin{aligned}t(u|q) \mathbf{Q}_1(u) &= \mathbf{Q}_1(u + 1) + q \cdot (u_1 u_2)^N \cdot \mathbf{Q}_1(u - 1), \\ t(u|q) \mathbf{Q}_2(u) &= q \cdot \mathbf{Q}_2(u + 1) + (u_1 u_2)^N \cdot \mathbf{Q}_2(u - 1).\end{aligned}\tag{5.34}$$

We also need the auxiliary transfer matrices involving \mathbf{r}^\pm (5.30), (5.31) of the form

$$\mathbf{q}_+ = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbf{r}_{10}^+(u_1 | u_2) \cdots \mathbf{r}_{N0}^+(u_1 | u_2) \right],\tag{5.35}$$

$$\mathbf{q}_- = \text{tr}_0 \left[q^{z_0 \partial_0} \mathbf{r}_{10}^-(u_1 | u_2) \cdots \mathbf{r}_{N0}^-(u_1 | u_2) \right].\tag{5.36}$$

For brevity the dependence on ℓ and on the parameter of regularization q is suppressed in the notation. Choosing the parameters in (5.22) in a special way we obtain

$$\frac{1}{1 - q} \cdot \mathbf{S} = \mathbf{q}_- \cdot \mathbf{q}_+ = \mathbf{q}_+ \cdot \mathbf{q}_-\tag{5.37}$$

In order to connect the Baxter operators \mathbf{Q}_1 and \mathbf{Q}_- we need the local factorization relation

$$\begin{aligned}\mathbb{R}_{00'}^1(v_1 | w_1, u_2) \cdot \mathbf{R}_{k0'}^1(u_1 | w_1, u_2) \cdot \mathbf{r}_{k0}^-(u_1 | u_2) &= \\ = \mathbf{S}_{k0} \cdot \mathbf{R}_{k0'}^-(u_1, u_2 | w_1) \cdot \mathbb{R}_{00'}^1(u_1 | w_1, u_2)\end{aligned}$$

It produces the corresponding relation for the transfer matrices. Thus we obtain

$$\mathbf{Q}_-(u) = \mathbf{S}^{-1} \cdot \mathbf{Q}_1(u) \cdot \mathbf{q}_-\tag{5.38}$$

and similarly

$$\mathbf{Q}_+(u) = \mathbf{S}^{-1} \cdot \mathbf{q}_+ \cdot \mathbf{Q}_2(u)\tag{5.39}$$

It is also possible to deduce commutativity of the operators \mathbf{q}_\pm , \mathbf{Q}_\pm , \mathbf{Q}_1 , \mathbf{Q}_2 and \mathbf{S} among themselves from local intertwining relations for their building blocks. As we have mentioned above such local relations underlying factorization and commutativity can be obtained from (4), (4.5), (4.6) restricted to finite-dimensional subspace. Further comments can be found in Appendix C.

The Baxter equations (5.34) for $\mathbf{Q}_{1,2}$ imply the ones for \mathbf{Q}_\pm

$$t(u|q) \mathbf{Q}_-(u) = \mathbf{Q}_-(u + 1) + q \cdot (u_1 u_2)^N \cdot \mathbf{Q}_-(u - 1),\tag{5.40}$$

$$t(u|q) \mathbf{Q}_+(u) = q \cdot \mathbf{Q}_+(u + 1) + (u_1 u_2)^N \cdot \mathbf{Q}_+(u - 1).\tag{5.41}$$

However they can also be obtained in the other way (see comments in Section 4.1), i.e. there are local relations which produce the Baxter equations.

5.6 Explicit action on polynomials

Now we proceed to calculate explicitly the action of the Baxter operators \mathbf{Q}_+ , \mathbf{Q}_- on polynomials. In Section 4.2 we have already computed the action of the Baxter operators \mathbf{Q}_+ , \mathbf{Q}_- constructed for infinite-dimensional representation to polynomials and have obtained explicit formulae. We are going to use these results and to take into account (5.26), i.e. we only have to consider the polynomials from appropriate subspace, apply the formulae from Section 4.2 and do the limit $\ell \rightarrow \frac{n}{2}$.

Thus we obtain immediately the explicit formula for the renormalized \mathbf{Q}_+ using (4.35)

$$\begin{aligned} \mathbf{Q}^+(u) : (1 - x_1 z_1)^n \cdots (1 - x_N z_N)^n &\mapsto \\ \mapsto \bar{\Pi}^n \cdot \prod_{k=1}^N \left(1 - \frac{x_k}{1-q} \cdot (z_{1,k-1} + q z_{k,N}) \right)^{\frac{n}{2}-u} \left(1 - \frac{x_k}{1-q} \cdot (z_{1,k} + q z_{k+1,N}) \right)^{\frac{n}{2}+u}, \end{aligned} \quad (5.42)$$

where

$$z_{1,k} \equiv z_1 + z_2 + \cdots + z_k \quad ; \quad z_{k,N} \equiv z_k + z_{k+1} + \cdots + z_N \quad ; \quad \bar{\Pi}^n \equiv \bar{\Pi}_1^n \bar{\Pi}_2^n \cdots \bar{\Pi}_N^n$$

$$\text{and } \bar{\Pi}_k^n \equiv \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(z_k \partial_k - n + \varepsilon)}{\Gamma(-n + \varepsilon)} = \frac{n! (-1)^{z_k \partial_k}}{\Gamma(1 + n - z_k \partial_k)} \cdot \Pi_k^n.$$

Then we turn to the second Baxter operator. We use (4.13) and rearrange there ε in such a way that the limits $\varepsilon \rightarrow 0$, $2\ell = n - \varepsilon$ exist for each operator factor,

$$\mathbf{Q}_-(u) = \varepsilon^N \mathbf{Q}_1(u|q) \cdot \frac{1}{\varepsilon^N} q^{-z_1 \partial_1} \mathbf{P}^{-1} \mathbf{q}_- \quad (5.43)$$

From (4.27), (4.29) we obtain

$$\frac{1}{\varepsilon^N} \cdot q^{-z_1 \partial_1} \mathbf{P}^{-1} \cdot \mathbf{q}_- \Psi(z_1, \dots, z_N) \rightarrow \Psi(z_2 - z_1, z_3 - z_2, \dots, q^{-1} z_1 - z_N) \quad (5.44)$$

where $\Psi(\vec{z})$ is a polynomial from the finite-dimensional subspace: $\Pi^n \Psi(\vec{z}) = \Psi(\vec{z})$.

In the limit $\varepsilon \rightarrow 0$ explicit the expression for operator $\mathbf{Q}_1(u|q)$ (4.31) becomes much simpler. In part I we have obtained

$$\varepsilon^N \cdot \mathbf{Q}_1(u|q) \Phi(\vec{z}) \rightarrow \frac{1}{\Gamma^N(1 + \frac{n}{2} - u) n!^N} \cdot \partial_{\lambda_1}^n \cdots \partial_{\lambda_N}^n \frac{(\bar{\lambda}_1 \cdots \bar{\lambda}_N)^{\frac{n}{2}-u}}{1 - q \bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Phi(\Lambda_q'^{-1} \vec{z}) \Big|_{\lambda=0} \quad (5.45)$$

where $\Phi(\vec{z})$ is an arbitrary polynomial and $\bar{\lambda} \equiv 1 - \lambda$.

Combining (5.44) with (5.45) and changing normalization we obtain finally the explicit formula

$$\mathbf{Q}^-(u) \Psi(\vec{z}) = \partial_{\lambda_1}^n \cdots \partial_{\lambda_N}^n \frac{(\bar{\lambda}_1 \cdots \bar{\lambda}_N)^{\frac{n}{2}-u}}{1 - q \bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \mathbf{M} \vec{z}) \Big|_{\lambda=0}. \quad (5.46)$$

where

$$\mathbf{M} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 1 \\ \frac{1}{q} & 0 & 0 & \dots & 0 & -1 \end{pmatrix}; \quad \Lambda_q' = \begin{pmatrix} 1 - \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & 1 - \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_3} & \frac{1}{\lambda_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 - \frac{1}{\lambda_{N-1}} & \frac{1}{\lambda_{N-1}} \\ \frac{1}{q \lambda_N} & 0 & 0 & 0 & \dots & 0 & 1 - \frac{1}{\lambda_N} \end{pmatrix}$$

6 Discussion

In two papers we have studied in great detail two variants of the construction of Q-operators in the framework of QISM [1–4]. We use the main ingredients of QISM, the L-matrices, the general R-matrices and the local commutation relations between them. As stated in Introduction the general scheme consists of two main steps:

First of all one needs the general or universal R-matrix – the general solution of Yang-Baxter equation. The general R-matrix can be factorized in a product of simpler operators which serve as building blocks for Q-operators.

The second step is the proof of the factorization of the general transfer matrix, constructed from the general R-matrix into the product of Q-operators. This factorization allows to derive the Baxter equation for Q-operators and various fusion relations.

We have unified two approaches by deducing all relations for transfer matrices and Baxter operators from local relations for their building blocks in the spirit of QISM. The local relations in turn are derived from three basic local relations of Yang-Baxter type. We consider this simplification of argumentation as one of the main achievements. It allows to reproduce the known results in much more transparent and systematical way.

Further, we have included in one scheme the treatment of infinite-dimensional representations in the quantum space as well as all finite-dimensional representations.

We believe that the main goal is not achieved by having done the pure algebraic construction of Q-operators as transfer matrices but includes also the explicit description of these operators. Therefore we have derived explicit formulae for the action of Q-operators on polynomials representing the quantum states of the chain.

Let us discuss now the particular results obtained in the present paper. Comparing the ordinary transfer matrix $t(u)$, its generalizations $t_n(u)$, $T_s(u)$ and the Baxter operators a simple systematics in their construction is evident. All they are constructed as traces of products of operators with one factor for each chain site including the q -regularization of the traces. The factor operators act on the tensor product of the quantum and the auxiliary spaces. In the construction with infinite-dimensional representations in the quantum space at generic spin ℓ in the most general case the factor at site k is the general Yang-Baxter operator \mathbb{R}_{k0} . In the other cases the factor operators are certain reductions obtained therefrom by imposing conditions on the representation parameters v_1, v_2 referring to the auxiliary space and/or extracting the asymptotics for representation parameters v_1, v_2 to infinity:

<i>chain operator</i>	<i>site operator</i>	<i>restriction</i>
T_s	\mathbb{R}_{k0}	—
t	$L_k \sim \mathbf{R}_{k0}(\ell, \frac{1}{2})$	$v_2 - v_1 = 2$ and Π_0^1
Q_+	\mathbb{R}_{k0}^+	$v_1 \rightarrow \infty$
Q_-	\mathbb{R}_{k0}^-	$v_2 \rightarrow \infty$

There are also the cases of twofold restrictions, where conditions are imposed on both v_1 and v_2 resulting in simpler operators:

<i>chain operator</i>	<i>site operator</i>	<i>restrictions</i>
q^+	\mathbb{r}_{k0}^+	$v_2 = u_2, v_1 \rightarrow \infty$
q^-	\mathbb{r}_{k0}^-	$v_1 = u_1, v_2 \rightarrow \infty$
$P q^{z_1 \partial_1}$	P_{k0}	$v_1 = u_1, v_2 = u_2$

The general transfer matrix T_s factorizes into the products of Q_+ and Q_- . The proofs of factorization and commutativity for different transfer matrices rely on local three-term relations — Yang-Baxter relations.

The construction in [9] is based on local relations including L-operators acting in the tensor product of the two-dimensional quantum space and the infinite-dimensional auxiliary space. We work with the general \mathbb{R} -operator and its reductions acting in the tensor product of two infinite-dimensional spaces $\mathbb{U}_{-\ell} \otimes \mathbb{U}_{-s}$ and formulate for them appropriate local relations. That means we perform the lifting to the level of \mathbb{R} -operators. In particular we have shown that their reduction to two-dimensional subspace in quantum space at $\ell = \frac{1}{2}$, i.e. descending to the level of L-operators, produces the local relation of [9]. This step of lifting provides the advantage of the unified treatment of the cases of generic and all finite-dimensional representations.

The systematics in the relation between global chain operators and local building blocks applies in analogy also to the case of integer or half-integer spin $\ell = \frac{n}{2}$ with finite-dimensional representation spaces at the sites. In the case of the general transfer matrix \mathbf{T}_s the site operators are now $\mathbf{R}_{k0}(u|\frac{n}{2}, s)$, the Yang-Baxter operators restricted to the irreducible subspace by means of the projector Π_k^n at $u_2 - u_1 = n + 1$. Additional restrictions on the parameters v_1, v_2 lead to the other reductions:

<i>chain operator</i>	<i>site operator</i>	<i>additional restriction</i>
\mathbf{T}_s	\mathbf{R}_{k0}	—
\mathbf{Q}_+	\mathbf{R}_{k0}^+	$v_1 \rightarrow \infty$
\mathbf{Q}_-	\mathbf{R}_{k0}^-	$v_2 \rightarrow \infty$

The presented proofs of factorization and commutativity of transfer matrices for finite-dimensional representations are completely parallel to the corresponding proofs of the infinite-dimensional case. Our analysis clarifies the relation between the cases of generic spins and half-integer or integer spins: by taking the limit $\ell \rightarrow \frac{n}{2}$ in the Baxter operators \mathbf{Q}_{\pm} we obtain immediately the appropriate set of Baxter operators \mathbf{Q}_{\pm} which do not map beyond the finite-dimensional subspace, i.e. Baxter operators \mathbf{Q}_{\pm} admit the restriction to the finite-dimensional subspace as well as the general transfer matrix \mathbf{T}_s does. The reason lies in the property shared by their building blocks \mathbb{R}^+ , \mathbb{R}^- , \mathbb{R} which all admit the restriction to finite-dimensional subspace in quantum space of the site.

Moreover, besides presenting general formulae for Baxter operators and proving their algebraic properties we present explicit compact expressions for their action on polynomials.

Another important achievement consists in establishing the relations between the present construction following the scheme of [9] to the construction considered in part I. There we have constructed Baxter operators $\mathbf{Q}_{1,2}$ for infinite-dimensional representations and Baxter operators $\mathbf{Q}_{1,2}$ for finite-dimensional representations and have proven the corresponding factorization formulae for the general transfer matrix. The unification of both constructions and the detailed comparison was the main aim of the present paper.

Let us start this discussion with the case of infinite-dimensional representations. We have shown that \mathbf{Q}_- and \mathbf{Q}_+ differ from \mathbf{Q}_1 and \mathbf{Q}_2 essentially by some dressing factors q_- or q_+ , which are roughly speaking inverse to each other. The operators q_{\pm} do not depend on spectral parameter and play a passive role. Indeed we have derived Baxter equations for \mathbf{Q}_{\pm} straightforwardly from the ones for $\mathbf{Q}_{1,2}$. That is both sets of Baxter operators $\mathbf{Q}_{1,2}$ and \mathbf{Q}_{\pm} contain identical information about the quantum system but the operators $\mathbf{Q}_{1,2}$ are simpler. Moreover, the essential shortcoming of the construction of \mathbf{Q}_{\pm} -operators is the unavoidable q -regularization violating sl_2 symmetry. It appears in the factorization relations $\mathbf{T} \sim \mathbf{Q}_+ \cdot \mathbf{Q}_-$ and in the expression for \mathbf{Q}_+ even in the infinite-dimensional case, whereas the trace in \mathbf{Q}_- converges for generic ℓ without q -regularization. The construction of Baxter operators \mathbf{Q}_1 and \mathbf{Q}_2 does not need the q -regularization in the infinite-dimensional case. It seems that the construction of the operators \mathbf{Q}_1 and \mathbf{Q}_2 is more adequate in the infinite-dimensional case.

In the case of finite-dimensional representation the situation is opposite. First of all in this case the dressing factor q_+ relating \mathbf{Q}_+ to \mathbf{Q}_2 reveals its significance: it ensures the invariance of

the finite-dimensional representation subspace under the action of Q_+ , whereas Q_2 does not have this property. For this reason the connection between the Baxter operators Q_\pm for generic spin and \mathbf{Q}_\pm for the ones for half-integer spin is simple: we can just put $\ell = \frac{n}{2}$ and the restriction to the irreducible subspace is straightforward. Comparing to part I we see that there the situation is different as far as the relation between generic and compact spin cases is concerned. We have emphasized there that the limit $2\ell \rightarrow n$ has to be performed with care in particular in the cases where Q_2 or the parameter restriction $v_1 = u_1$ is involved. Because in the limit to integer values of 2ℓ complications are absent in the case of Q_\pm this construction appears more adequate in the finite-dimensional representation case. To summarize, both pairs of Q operators are complementary in some sense and each of them has its favourable case of representations.

In both papers we have restricted ourselves to the simplest example of the spin chain with the symmetry algebra of rank one. There are a plenty of results in more general situations. The approach of [8] and [9] is generalized in [14, 15] to the cases of algebras of higher rank and in [16, 17] to the case of superalgebras. The another approach is also applicable in these situations: see [18–20] for $SL(n, \mathbb{C})$ and [21] for superalgebras.

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Appendices

A Reductions of RLL-relations

Here we consider the calculation of reductions in RLL-relations and derive explicit expressions for operators r^+ and R^+ .

A.1 Reduction $R^1 \rightarrow r^+$

We start from the defining equation for operator R^1 (3.8) and take the limit $v_1 \rightarrow \infty$ using (3.5)

$$R^1(u_1|v_1, v_2) L_1(u_1, u_2) L_2^-(v_2) \begin{pmatrix} v_1 & 0 \\ 0 & 1 \end{pmatrix} = L_1^-(u_2) \begin{pmatrix} v_1 & 0 \\ 0 & 1 \end{pmatrix} L_2(u_1, v_2) R^1(u_1|v_1, v_2)$$

or in equivalent form

$$R^1(u_1|v_1, v_2) L_1(u_1, u_2) L_2^-(v_2) = L_1^-(u_2) \begin{pmatrix} v_1 & 0 \\ 0 & 1 \end{pmatrix} L_2(u_1, v_2) \begin{pmatrix} v_1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} R^1(u_1|v_1, v_2) .$$

The $s\ell_2$ -invariance of the L-operator allows to transform the matrix similarity transformation to the operator similarity transformation,

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} L(u) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \lambda^{-z\partial} L(u) \lambda^{z\partial} . \quad (A.1)$$

Thus we obtain relation of the form (3.12)

$$v_1^{z_2 \partial_2} R^1(u_1|v_1, v_2) \cdot L_1(u_1, u_2) L_2^-(v_2) = L_1^-(u_2) L_2(u_1, v_2) \cdot v_1^{z_2 \partial_2} R^1(u_1|v_1, v_2) . \quad (\text{A.2})$$

Now we are going to calculate the leading term of $v_1^{z_2 \partial_2} R^1(u_1|v_1, v_2)$ in asymptotics $v_1 \rightarrow \infty$

$$\begin{aligned} v_1^{z_2 \partial_2} R^1(u_1|v_1, v_2) &= v_1^{z_2 \partial_2} \cdot \frac{\Gamma(z_{21} \partial_2 + u_1 - v_2 + 1)}{\Gamma(z_{21} \partial_2 + v_1 - v_2 + 1)} = v_1^{z_2 \partial_2} \cdot e^{-z_1 \partial_2} \frac{\Gamma(z_2 \partial_2 + u_1 - v_2 + 1)}{\Gamma(z_2 \partial_2 + v_1 - v_2 + 1)} e^{z_1 \partial_2} = \\ &= e^{-\frac{z_1 \partial_2}{v_1}} \frac{v_1^{z_2 \partial_2} \cdot \Gamma(z_2 \partial_2 + u_1 - v_2 + 1)}{\Gamma(z_2 \partial_2 + v_1 - v_2 + 1)} e^{z_1 \partial_2} \longrightarrow (2\pi)^{\frac{1}{2}} e^{v_1} v_1^{v_2 - v_1 - \frac{1}{2}} \cdot \Gamma(z_2 \partial_2 + u_1 - v_2 + 1) e^{z_1 \partial_2} \end{aligned}$$

Above we use Stirling's formula for the asymptotics of the Γ -function

$$\Gamma(\lambda + a) \xrightarrow{\lambda \rightarrow \infty} (2\pi)^{\frac{1}{2}} e^{-\lambda} \lambda^{\lambda - \frac{1}{2}} \cdot \lambda^a. \quad (\text{A.3})$$

Note that the factor $(2\pi)^{-\frac{1}{2}} e^{v_1} v_1^{v_2 - v_1 - \frac{1}{2}}$ can be removed from both sides of equation (A.2)⁴ and we obtain the operator r^+ (3.13). Thus we have specified the reduction procedure for the operator R^1 starting from our prescription for the reduction of the L-operator and using the RLL relation entangling them. That is, the reduction procedure for L-operators determines the procedure for R-operators.

Similar calculations for the operator R^2 leads to (3.14) and (3.15).

A.1.1 Reduction $R \rightarrow R^+$

Since the general R-operator is constructed from the operators R^1 and R^2 its reduction follows from the building block reduction considered above. We start from the defining equation for R-operator (3.6) and take the limit $v_1 \rightarrow \infty$. Just in the same way as before applying (A.1) we deduce

$$v_1^{z_2 \partial_2} R(u_1, u_2|v_1, v_2) \cdot L_1(u_1, u_2) L_2^-(v_2) = L_1^-(v_2) L_2(u_1, u_2) \cdot v_1^{z_2 \partial_2} R(u_1, u_2|v_1, v_2). \quad (\text{A.4})$$

Calculating the leading term of

$$v_1^{z_2 \partial_2} R(u_1, u_2|v_1, v_2) = v_1^{z_2 \partial_2} R^1(u_1|v_1, u_2) R^2(u_1, u_2|v_2)$$

in asymptotic $v_1 \rightarrow \infty$ we see that the operator $R^2(u_1, u_2|v_2)$ does not depend on v_1 and therefore it remains unchanged in the limit. So the calculation of the previous section leads to the operator R^+ (3.17)⁵ which interchanges L^- and L (3.16). Repeating the same procedure at $v_2 \rightarrow \infty$ we obtain (3.18) and (3.19).

A.1.2 Double reductions

Above we have implemented one reduction in the RLL relations excluding parameters by taking appropriate limits. Now we perform one more reduction in the RLL relations.

We take the limit $v_2 \rightarrow \infty$ in (3.16) and obtain

$$v_2^{-z_2 \partial_2} R^+(u_1, u_2|v_2) \cdot L_1(u_1, u_2) \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} L_2(u_1, u_2) \cdot v_2^{-z_2 \partial_2} R^+(u_1, u_2|v_2)$$

⁴In the following we will omit such c-number factor and the notation $v_1^{z_2 \partial_2} R^1(u_1|v_1, v_2) \rightarrow r^+(u_1|v_2)$ implies it.

⁵The notation R^+ and similarly for the reductions of the R-operators are chosen in such a way that after restriction to finite dimensional subspace we obtain correspondingly L^+ .

Let us calculate the limit explicitly. At first we rewrite $R^+(u_1, u_2|v_2)$ taking into account (3.17), (3.13), (3.10) and the useful representation for the permutation, $P_{12} = e^{-z_2\partial_1} e^{z_1\partial_2} e^{-z_2\partial_1} (-)^{z_1\partial_1}$,

$$R^+(u_1, u_2|v_2) = P_{12} \Gamma(z_1\partial_1 + u_1 - u_2 + 1) e^{z_1\partial_2} (-)^{z_1\partial_1} \frac{\Gamma(z_1\partial_1 + u_1 - v_2 + 1)}{\Gamma(z_1\partial_1 + u_1 - u_2 + 1)} e^{z_2\partial_1}.$$

Thus using (A.3) one readily obtains

$$v_2^{-z_2\partial_2} R^+(u_1, u_2|v_2) \rightarrow P_{12} e^{z_2\partial_1} \quad (\text{A.5})$$

Then we calculate the limit $v_2 \rightarrow \infty$ in (3.12) using (3.13)

$$v_2^{-z_2\partial_2} r^+(u_1|v_2) \cdot L_1(u_1, u_2) \cdot \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} = L_1^-(u_2) L_2^+(u_1) \cdot v_2^{-z_2\partial_2} r^+(u_1|v_2)$$

$$v_2^{-z_2\partial_2} r^+(u_1|v_2) \rightarrow (-)^{z_2\partial_2} e^{z_1\partial_2} \quad (\text{A.6})$$

It is also possible to calculate the reduction of (3.16) at $u_1 \rightarrow \infty$

$$R^{++}(u_2|v_2) \cdot L_1^-(u_2) L_2^-(v_2) = L_1^-(v_2) L_2^-(u_2) \cdot R^{++}(u_2|v_2) \quad (\text{A.7})$$

$$R^+(u_1, u_2|v_2) u_1^{-z_2\partial_2} \rightarrow R^{++}(u_2|v_2) = (1 - z_2\partial_1)^{u_2-v_2} \quad (\text{A.8})$$

and the reduction of (3.18) at $u_2 \rightarrow \infty$

$$R^{--}(u_1|v_1) \cdot L_1^+(u_1) L_2^+(v_1) = L_1^+(v_1) L_2^+(u_1) \cdot R^{--}(u_1|v_1) \quad (\text{A.9})$$

$$u_2^{z_2\partial_2} R^-(u_1, u_2|v_1) \rightarrow R^{--}(u_1|v_1) = (1 + z_1\partial_2)^{u_1-v_1} \quad (\text{A.10})$$

using similar methods as before.

B Local relations

Here we shall derive the local factorization and commutativity relations which underlie the factorization of the general transfer matrix and its commutativity.

B.1 Local factorization

We start with the derivation of (4.12) which is the local variant of (4.10). In (4.5) we choose the first space to be the local quantum space $\mathbb{U}_{-\ell}$ in site k , the second space to be the auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_0]$ and the third space to be another copy of the auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_{0'}]$

$$\begin{aligned} \mathbb{R}_{00'}^1(v_1|w_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{k0}(u_1, u_2|v_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{00'}^1(v_1|w_1, w_2) \end{aligned} \quad (\text{B.1})$$

We have to consider the appropriate limiting procedure. We multiply this relation by the dilatation operator $w_1^{z_0\partial_0 + z_k\partial_k}$ from the left and use

$$[z_0\partial_0 + z_k\partial_k, R_{k0}] = 0 \quad (\text{B.2})$$

to obtain⁶

$$w_1^{z_0\partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w_1}, w_2) \cdot w_1^{z_k\partial_k} \mathbb{R}_{k0'}(u_1, u_2|\underline{w_1}, w_2) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) =$$

⁶Here we underline parameters to be excluded by taking asymptotics.

$$= \mathbb{R}_{k0}(u_1, u_2|v_1, w_2) \cdot w_1^{z_k \partial_k} \mathbb{R}_{k0'}(u_1, u_2|\underline{w_1}, v_2) \cdot w_1^{z_0 \partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w_1}, w_2)$$

Then we take limit $w_1 \rightarrow \infty$ with the help of (3.13), (3.17)

$$\begin{aligned} & \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|\underline{w_2}) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, \underline{w_2}) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|v_2) \cdot \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \end{aligned}$$

At the next step we multiply by $w_2^{-z_0 \partial_0 - z_k \partial_k}$ from the left and use (B.2) again

$$\begin{aligned} & w_2^{-z_0 \partial_0} \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \cdot w_2^{-z_k \partial_k} \mathbb{R}_{k0'}^+(u_1, u_2|\underline{w_2}) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, \underline{w_2}) w_2^{-z_k \partial_k} \cdot \mathbb{R}_{k0'}^+(u_1, u_2|v_2) \cdot w_2^{-z_0 \partial_0} \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \end{aligned}$$

Then we take the limit $w_2 \rightarrow \infty$ exploiting (A.6), (A.5), (3.19) and finally obtain (4.12)

$$\begin{aligned} & P_{00'}(-)^{z_{0'} \partial_{0'}} e^{z_0 \partial_{0'}} \cdot e^{z_{0'} \partial_k} \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}^-(u_1, u_2|v_1) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|v_2) \cdot P_{00'}(-)^{z_{0'} \partial_{0'}} e^{z_0 \partial_{0'}} \end{aligned}$$

Similarly form (4.6) one can obtain the local factorization relation

$$\begin{aligned} & \mathbb{r}_{00'}^-(v_1|v_2) \cdot P_{k0'} \mathbb{r}_{k0'}^+(u_1|u_2) \mathbb{r}_{k0'}^-(u_1|u_2) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}^+(u_1, u_2|v_2) \cdot \mathbb{R}_{k0'}^-(u_1, u_2|v_1) \cdot \mathbb{r}_{00'}^-(v_1|v_2) \end{aligned} \tag{B.3}$$

which implies the first factorization in (4.10) if we take into account that

$$P_{12} \mathbb{r}^+(u_1|u_2) \mathbb{r}^-(u_1|u_2) = \Gamma(z_1 \partial_1 + u_1 - u_2 + 1) e^{z_1 \partial_2} \Gamma^{-1}(z_1 \partial_1 + u_1 - u_2 + 1)$$

Then we derive the local factorization relation underlying (4.13). In the relation (B.1) we choose $w_2 = u_2, v_1 = u_1$

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, u_2) \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{k0}^2(u_1, u_2|\underline{v_2}) = \\ & = P_{k0} \mathbb{R}_{k0'}(u_1, u_2|w_1, \underline{v_2}) \mathbb{R}_{00'}^1(u_1|w_1, u_2) \end{aligned}$$

then multiply by the dilatation operator $v_2^{-z_k \partial_k}$ from the right, take the limit $v_2 \rightarrow \infty$ using (3.15) and obtain the needed relation (3.19)

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, u_2) \cdot \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \cdot \mathbb{r}_{k0}^-(u_1|u_2) = \\ & = P_{k0} \cdot \mathbb{R}_{k0'}^-(u_1, u_2|w_1) \cdot \mathbb{R}_{00'}^1(u_1|w_1, u_2). \end{aligned}$$

Similarly (4.6) results in

$$\begin{aligned} & \mathbb{R}_{00'}^2(u_1, v_2|u_2) \cdot \mathbb{r}_{k0'}^+(u_1|u_2) \cdot \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^-(u_1, u_2|v_2) \cdot P_{k0'} \cdot \mathbb{R}_{00'}^2(u_1, v_2|u_2) \end{aligned}$$

implying the factorization (4.14).

B.2 Local commutativity

Now we are going to present the local relations which underlie commutativity relations for transfer matrices. For this purpose we use the general Yang-Baxter relation (4). Rewriting it in the form

$$\begin{aligned} & \mathbb{R}_{00'}(v_1, v_2|w_1, w_2)\mathbb{R}_{k0'}(u_1, u_2|w_1, w_2)\mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, v_2)\mathbb{R}_{k0'}(u_1, u_2|w_1, w_2)\mathbb{R}_{00'}(v_1, v_2|w_1, w_2) \end{aligned} \quad (\text{B.4})$$

we derive immediately the commutativity of the general transfer matrices (4.1). The commutativity for the other transfer matrices follows from

$$\begin{aligned} & \mathbb{R}_{00'}(u_1, v_2|w_1, u_2)\mathbb{R}_{k0'}^1(u_1|w_1, u_2)\mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2)\mathbb{R}_{k0'}^1(u_1|w_1, u_2)\mathbb{R}_{00'}(u_1, v_2|w_1, u_2) \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, u_2)\mathbb{R}_{k0'}^1(u_1|w_1, u_2)\mathbb{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbb{R}_{k0}^1(u_1|v_1, u_2)\mathbb{R}_{k0'}^1(u_1|w_1, u_2)\mathbb{R}_{00'}^1(v_1|w_1, u_2) \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} & \mathbb{R}_{00'}^2(u_1, v_2|w_2)\mathbb{R}_{k0'}^2(u_1, u_2|w_2)\mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2)\mathbb{R}_{k0'}^2(u_1, u_2|w_2)\mathbb{R}_{00'}^2(u_1, v_2|w_2) \end{aligned} \quad (\text{B.7})$$

which are obtained from (B.2) specifying parameters (for more details see part I, Appendix A).

We start with (B.6) multiply it by $w_1^{z_0\partial_0+z_k\partial_k}$ from the left

$$\begin{aligned} & w_1^{z_0\partial_0}\mathbb{R}_{00'}^1(v_1|\underline{w_1}, u_2) \cdot w_1^{z_k\partial_k}\mathbb{R}_{k0'}^1(u_1|\underline{w_1}, u_2) \cdot \mathbb{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbb{R}_{k0}^1(u_1|v_1, u_2) \cdot w_1^{z_k\partial_k}\mathbb{R}_{k0'}^1(u_1|\underline{w_1}, u_2) \cdot w_1^{z_0\partial_0}\mathbb{R}_{00'}^1(v_1|\underline{w_1}, u_2) \end{aligned}$$

and do the limit $w_1 \rightarrow \infty$ by using (3.13)

$$\begin{aligned} & \mathbb{r}_{00'}^+(v_1|u_2) \cdot \mathbb{r}_{k0'}^+(u_1|u_2) \cdot \mathbb{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbb{R}_{k0}^1(u_1|v_1, u_2) \cdot \mathbb{r}_{k0'}^+(u_1|u_2) \cdot \mathbb{r}_{00'}^+(v_1|u_2). \end{aligned} \quad (\text{B.8})$$

The last relation implies the commutativity of the transfer matrices constructed from \mathbb{r}^+ and \mathbb{R}^1 . In much the same way (B.7) produces

$$\begin{aligned} & \mathbb{r}_{00'}^-(u_1|v_2) \cdot \mathbb{r}_{k0'}^-(u_1|u_2) \cdot \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \cdot \mathbb{r}_{k0'}^-(u_1|u_2) \cdot \mathbb{r}_{00'}^-(u_1|v_2) \end{aligned} \quad (\text{B.9})$$

resulting in the commutativity of the transfer-matrices built from \mathbb{r}^- and \mathbb{R}^2 .

In order to obtain the other commutation relations we use (B.5) multiply it by $w_1^{z_0\partial_0+z_k\partial_k}$ from the left

$$\begin{aligned} & w_1^{z_0\partial_0}\mathbb{R}_{00'}(u_1, v_2|\underline{w_1}, u_2) \cdot w_1^{z_k\partial_k}\mathbb{R}_{k0'}^1(u_1|\underline{w_1}, u_2) \cdot \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \cdot w_1^{z_k\partial_k}\mathbb{R}_{k0'}^1(u_1|\underline{w_1}, u_2) \cdot w_1^{z_0\partial_0}\mathbb{R}_{00'}(u_1, v_2|\underline{w_1}, u_2) \end{aligned}$$

and take the limit $w_1 \rightarrow \infty$ by means of (3.13), (3.17)

$$\begin{aligned} & \mathbb{R}_{00'}^+(u_1, v_2|u_2) \cdot \mathbb{r}_{k0'}^+(u_1|u_2) \cdot \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \cdot \mathbb{r}_{k0'}^+(u_1|u_2) \cdot \mathbb{R}_{00'}^+(u_1, v_2|u_2). \end{aligned} \quad (\text{B.10})$$

This means that the transfer matrices constructed from \mathfrak{r}^+ and \mathbb{R}^2 commute. Similarly it is easy to derive the intertwining relation for \mathfrak{r}^- and \mathbb{R}^1

$$\begin{aligned} & \mathbb{R}_{00'}^-(v_1, u_2|u_1) \cdot \mathfrak{r}_{k0'}^-(u_1|u_2) \cdot \mathbb{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbb{R}_{k0}^1(u_1|v_1, u_2) \cdot \mathfrak{r}_{k0'}^-(u_1|u_2) \cdot \mathbb{R}_{00'}^-(v_1, u_2|u_1). \end{aligned} \quad (\text{B.11})$$

In principle combining commutativity relations (produced by local relations presented above) with factorization relations for transfer matrices we deduce immediately the commutativity for the all transfer matrices. However it is also possible to prove the commutativity relations from the local intertwining relations

$$\begin{aligned} & \mathbb{R}_{00'}^+(v_1, v_2|w_2) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|w_2) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|w_2) \cdot \mathbb{R}_{00'}^+(v_1, v_2|w_2). \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} & \mathbb{R}_{00'}^{++}(v_2|w_2) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|w_2) \cdot \mathbb{R}_{k0}^+(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^+(u_1, u_2|v_2) \cdot \mathbb{R}_{k0'}^+(u_1, u_2|w_2) \cdot \mathbb{R}_{00'}^{++}(v_2|w_2) \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} & \mathbb{R}_{00'}^-(v_1, v_2|w_1) \cdot \mathbb{R}_{k0'}^-(u_1, u_2|w_1) \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) \cdot \mathbb{R}_{k0'}^-(u_1, u_2|w_1) \cdot \mathbb{R}_{00'}^-(v_1, v_2|w_1). \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} & \mathbb{R}_{00'}^{--}(v_1|w_1) \cdot \mathbb{R}_{k0'}^-(u_1, u_2|w_1) \cdot \mathbb{R}_{k0}^-(u_1, u_2|v_1) = \\ & = \mathbb{R}_{k0}^-(u_1, u_2|v_1) \cdot \mathbb{R}_{k0'}^-(u_1, u_2|w_1) \cdot \mathbb{R}_{00'}^{--}(v_1|w_1). \end{aligned} \quad (\text{B.15})$$

which can be obtained from (B.2) performing appropriate limiting procedures in much the same way as before.

C Finite-dimensional construction

Here we perform some calculations which concern the Baxter operator construction for finite-dimensional representations relying on part I, section 5.

C.1 Restriction of the general \mathbb{R} -operator to finite-dimensional representations

In part I, subsection 5.2, we have obtained as the explicit form of the restricted \mathbb{R} -operator

$$\begin{aligned} \mathbf{R}_{12}(u|\frac{n}{2}, s) &= P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2}}{\Gamma(z_2 \partial_2 - \frac{n}{2} - s - u) \Gamma(1 + n - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1} \Gamma(z_1 \partial_1 - \frac{n}{2} - s + u) \Gamma(1 + n - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n. \end{aligned} \quad (\text{C.1})$$

Recall also that finite-dimensional subspace is invariant under the action of operator $\mathbf{R}_{12}(u|\frac{n}{2}, s)$ because projector Π_1^n appears not only on the right but on the left too

$$P_{12} e^{-z_1 \partial_2} \Pi_2^n = P_{12} \Pi_2^n e^{-z_1 \partial_2} \Pi_2^n = \Pi_1^n P_{12} e^{-z_1 \partial_2} \Pi_2^n. \quad (\text{C.2})$$

In the parametrization (5.16)

$$u_1 = u - \frac{n}{2} - 1, \quad u_2 = u + \frac{n}{2} \quad ; \quad v_1 = v - s - 1, \quad v_2 = v + s$$

we rewrite (C.1) as follows

$$\begin{aligned} \mathbf{R}_{12}(u_1, u_2|v_1, v_2) &= P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1} \Gamma(z_1 \partial_1 + u_1 - v_2 + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n. \end{aligned} \quad (\text{C.3})$$

Doing similar calculations as with the operators \mathbb{R}^+ and \mathbb{R}^- considered in subsection 3.2 we obtain

- Operator \mathbf{R}^+

$$\begin{aligned} \mathbf{R}_{12}^+(u_1, u_2|v_2) &= P_{12} \cdot \frac{(-1)^{z_2 \partial_2}}{\Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1} \Gamma(z_1 \partial_1 + u_1 - v_2 + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n \end{aligned} \quad (\text{C.4})$$

- Operator \mathbf{R}^-

$$\begin{aligned} \mathbf{R}_{12}^-(u_1, u_2|v_1) &= P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1} \Gamma(u_2 - u_1 - z_1 \partial_1) \Pi_1^n \end{aligned} \quad (\text{C.5})$$

One can easily see that \mathbf{R}^- and \mathbf{R}^+ do not map beyond the subspace $\mathbb{V}_n \otimes \mathbb{U}_{-s}$ as well as \mathbf{R} . The same is true for \mathbf{R}^1 , \mathbf{R}^2 , \mathbf{r}^+ and \mathbf{r}^- . In all these operators the parameters are as in (5.16).

Previously we have calculated the double reduction of R-operator (C.6). The finite-dimensional analogue of this formula is

$$v_2^{-z_1 \partial_1} \mathbf{R}^+(u_1, u_2|v_2) \rightarrow P_{12} e^{z_2 \partial_1} \Pi_1^n \quad (\text{C.6})$$

- Operator \mathbf{R}^1

Explicit expressions for \mathbf{R}^1 , \mathbf{R}^2 are obtained in part I:

$$\mathbf{R}_{12}^1(u_1|v_1, u_2) \equiv P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2 + u_2 - u_1 - 1}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \Pi_1^n \quad (\text{C.7})$$

- Operator \mathbf{R}^2

$$\begin{aligned} \mathbf{R}_{12}^2(u_1, u_2|v_2) &\equiv P_{12} \cdot e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1 + u_2 - u_1 - 1} \Gamma(z_1 \partial_1 + u_1 - v_2 + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n \end{aligned} \quad (\text{C.8})$$

Comparing expressions of \mathbf{R}^1 with \mathbf{R}^- and of \mathbf{R}^2 with \mathbf{R}^+ we that they are very similar. Indeed they are all derived from the same \mathbf{R} -operator.

- Operator \mathbf{r}^+

In order to calculate \mathbf{r}^+ , \mathbf{r}^- we start from definitions (5.30), (5.31) and use Stirling's formula (A.3)

$$\mathbf{r}_{12}^+(u_1|u_2) = P_{12} \cdot \frac{(-1)^{z_2 \partial_2 + u_2 - u_1 - 1}}{\Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \Pi_1^n \quad (\text{C.9})$$

- Operator \mathbf{r}^-

$$\mathbf{r}_{12}^-(u_1|u_2) = P_{12} \cdot e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \cdot e^{-z_2 \partial_1} (-1)^{u_2 - u_1 - 1} \Gamma(u_2 - u_1 - z_1 \partial_1) \Pi_1^n \quad (\text{C.10})$$

C.2 Local factorization

The derivation of the local relations in the case of finite-dimensional operators goes parallel to the calculation in the infinite-dimensional case. Let us obtain (5.21). In (4.5) we choose the first space to be the local quantum space $\mathbb{U}_{-\ell}$ in k -site, the second space to be the auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_0]$ and the third space to be another auxiliary space $\mathbb{U}_{-s} \sim \mathbb{C}[z_{0'}]$ and then restrict the quantum space to the finite-dimensional subspace \mathbb{V}_n at $\ell = \frac{n}{2}$

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|v_1, w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, v_2) \mathbb{R}_{00'}^1(v_1|w_1, w_2) \end{aligned}$$

We have to consider the appropriate limiting procedure. We multiply this relation by the dilatation operator $w_1^{z_0\partial_0+z_k\partial_k}$ from the left to obtain

$$\begin{aligned} & w_1^{z_0\partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w_1}, w_2) \cdot w_1^{z_k\partial_k} \mathbf{R}_{k0'}(u_1, u_2|\underline{w_1}, w_2) \cdot \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|v_1, w_2) \cdot w_1^{z_k\partial_k} \mathbf{R}_{k0'}(u_1, u_2|\underline{w_1}, v_2) \cdot w_1^{z_0\partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w_1}, w_2) \end{aligned}$$

Then we take the limit $w_1 \rightarrow \infty$ with the help of (3.13), (5.17)

$$\begin{aligned} & \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \cdot \mathbf{R}_{k0'}^+(u_1, u_2|\underline{w_2}) \cdot \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|v_1, \underline{w_2}) \cdot \mathbf{R}_{k0'}^+(u_1, u_2|v_2) \cdot \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \end{aligned}$$

At the next step we multiply by $w_2^{-z_0\partial_0-z_k\partial_k}$ from the left

$$\begin{aligned} & w_2^{-z_0\partial_0} \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \cdot w_2^{-z_k\partial_k} \mathbf{R}_{k0'}^+(u_1, u_2|\underline{w_2}) \cdot \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|v_1, \underline{w_2}) w_2^{-z_k\partial_k} \cdot \mathbf{R}_{k0'}^+(u_1, u_2|v_2) \cdot w_2^{-z_0\partial_0} \mathbb{r}_{00'}^+(v_1|\underline{w_2}) \end{aligned}$$

Then we take the limit $w_2 \rightarrow \infty$ using (A.6), (C.6), (5.18) and finally obtain (5.21)

$$\begin{aligned} & P_{00'}(-)^{z_{0'}\partial_{0'}} e^{z_0\partial_0} \cdot e^{z_{0'}\partial_k} \Pi_k^n \cdot \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}^-(u_1, u_2|v_1) \cdot \mathbf{R}_{k0'}^+(u_1, u_2|v_2) \cdot P_{00'}(-)^{z_{0'}\partial_{0'}} e^{z_0\partial_0} \end{aligned}$$

C.3 Local commutativity

In order to illustrate the derivation of the local commutativity relation we are going to derive the underlying commutativity of the transfer matrices \mathbf{r}^+ and \mathbf{R}^1 . We start with the general Yang-Baxter equation restricted to the space $\mathbb{V}^n \otimes \mathbb{C}[z_0] \otimes \mathbb{C}[z_{0'}]$

$$\begin{aligned} & \mathbb{R}_{00'}(v_1, v_2|w_1, w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{00'}(v_1, v_2|w_1, w_2) \end{aligned}$$

choose parameters $w_2 = u_2 - \delta$, $v_2 = u_2 - \delta$ and take the limit $\delta \rightarrow 0$ (5.27)

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, u_2) \mathbf{R}_{k0'}^1(u_1|w_1, u_2) \mathbf{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbf{R}_{k0}^1(u_1|v_1, u_2) \mathbf{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{00'}^1(v_1|w_1, u_2) \end{aligned} \tag{C.11}$$

Then we multiply it by $w_1^{z_0\partial_0+z_k\partial_k}$ from the left

$$w_1^{z_0\partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w_1}, u_2) \cdot w_1^{z_k\partial_k} \mathbf{R}_{k0'}^1(u_1|\underline{w_1}, u_2) \cdot \mathbf{R}_{k0}^1(u_1|v_1, u_2) =$$

$$= \mathbf{R}_{k0}^1(u_1|v_1, u_2) \cdot w_1^{z_k \partial_k} \mathbf{R}_{k0'}^1(u_1|\underline{w}_1, u_2) \cdot w_1^{z_0 \partial_0} \mathbb{R}_{00'}^1(v_1|\underline{w}_1, u_2)$$

and do the limit $w_1 \rightarrow \infty$ using (3.13), (5.30)

$$\begin{aligned} & \mathfrak{r}_{00'}^+(v_1|u_2) \cdot \mathbf{r}_{k0'}^+(u_1|u_2) \cdot \mathbf{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbf{R}_{k0}^1(u_1|v_1, u_2) \cdot \mathbf{r}_{k0'}^+(u_1|u_2) \cdot \mathfrak{r}_{00'}^+(v_1|u_2). \end{aligned}$$

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